SUBSEQUENCE SUMS OF ZERO-SUM FREE SEQUENCES OVER FINITE ABELIAN GROUPS

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ABSTRACT. Let G be a finite abelian group of rank r and let X be a zero-sum free sequence over G whose support $\mathrm{supp}(X)$ generates G. In 2009, Pixton proved that $|\Sigma(X)| \ge 2^{r-1}(|X| - r + 2) - 1$ for $r \le 3$. In this paper, we show that this result also holds for abelian groups G of rank 4 if the smallest prime p dividing $|G|$ satisfies $p \ge 13$.

1. INTRODUCTION

Let G be a finite abelian group. By a sequence over G we mean a finite sequence of terms from G where repetition is allowed and the order is disregarded. In 1972, Eggleton and Erdős[1] first tackled the problem of determining the minimal cardinality of the set $\Sigma(X)$ of subsums of zero-sum free sequences over G. In 1977, Olson and White[6] obtained a lower bound for $|\Sigma(X)|$, where X is a zero-sum free sequence over G and $\mathrm{supp}(X)$ generates a noncyclic group. Subsequently, several authors [2, 7, 8, 9] obtained a huge variety of results. In particular, we refer the reader to Part II of the recent monograph [5] by Grynkiewicz. In 2009, Pixton [7, Lemma 1.1, Theorem 1.3 and Theorem 1.7] proved the following theorem

Theorem A. *Let G be a finite abelian group of rank $r \le 3$, and let X be a zero-sum free sequence over G whose support generates G. Then $|\Sigma(X)| \ge 2^{r-1}(|X| - r + 2) - 1$ where $|X|$ denotes the length of X.*

In this paper, we show that the bound given in Theorem A also holds for a class of abelian groups of rank 4.

Main Theorem 1.1. *Let G be a finite abelian group of rank 4 and let X be a zero-sum free sequence over G whose support generates G. If the smallest prime p dividing $|G|$ satisfies $p \ge 13$, then $|\Sigma(X)| \ge 8|X| - 17$.*

Let G be a nontrivial finite abelian group, say $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$. Then $r = \mathsf{r}(G)$ is the rank of G.

Suppose that $r = 4$. Then the lower bound given in Theorem 1.1 is best possible. Indeed, if (e_1, e_2, e_3, e_4) is a basis of G, then the sequence $S = e_1e_2e_3e_4^{\mathrm{ord}(e_4)-1}$

2010 *Mathematics Subject Classification.* Primary 11B50; Secondary 11P99.
Key words and phrases. zero-sum free sequence, finite abelian group, Davenport's constant.

is zero-sum free, $|S| = \mathrm{ord}(e_4) + 2$, and $|\Sigma(S)| = 2^3\,\mathrm{ord}(e_4) - 1 = 8|S| - 17$. We do not know whether the result holds true for groups G having a prime divisor q which is smaller than 13. Note that we have no information on the maximal length or on the structure of zero-sum free sequences over groups of rank 4. We only want to recall that there are zero-sum free sequences S over groups of rank 4 whose lengths are strictly larger than $\sum_{i=1}^{4}(n_i - 1)$ (see [4]).

2. Preliminaries

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{Z}$ denote the set of integers. For $a, b \in \mathbb{Z}$ with $a \le b$, we define $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. Let G be an additively written finite abelian group. Let $\mathscr{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathscr{F}(G)$ are called *sequences* over G.

We write a sequence $S \in \mathscr{F}(G)$ in the form

$$S = \prod_{g \in G} g^{\mathsf{v}_g(S)}, \quad \text{with } \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S. We say that S *contains* g if $\mathsf{v}_g(S) > 0$. The unit element $1 \in \mathscr{F}(G)$ is called the *empty sequence*. A sequence S_1 is called a *subsequence* of S if $S_1 \mid S$ in $\mathscr{F}(G)$ (equivalently, $\mathsf{v}_g(S_1) \le \mathsf{v}_g(S)$ for all $g \in G$). Let $S_1, S_2 \in \mathscr{F}(G)$, we denote by $S_1 S_2$ the sequence

$$\prod_{g \in G} g^{\mathsf{v}_g(S_1) + \mathsf{v}_g(S_2)} \in \mathscr{F}(G),$$

and we denote by $S_1 S_2^{-1}$ the sequence

$$\prod_{g \in G} g^{\mathsf{v}_g(S_1) - \min\{\mathsf{v}_g(S_1), \mathsf{v}_g(S_2)\}} \in \mathscr{F}(G).$$

If a sequence $S \in \mathscr{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$. For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathscr{F}(G),$$

we call

- $|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S,
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S,
- $\mathrm{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subseteq G$ the *support* of S,
- $\Sigma(S) = \{\sum_{i \in I} g_i \mid \emptyset \ne I \subseteq [1, n]\}$ the *set of sub(sequence) sums* of S.

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *zero-sum free sequence* if $0 \notin \Sigma(S)$,
- a *minimal zero-sum sequence* if $|S| \ge 1$, $\sigma(S) = 0$, and S contains no proper and nontrivial zero-sum subsequence.

If G_1 is a group and $\varphi : G \to G_1$ a map, then $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$ is a sequence over G_1.

We denote by $\mathsf{D}(G) = \max\{|S| \mid S$ is a minimal zero-sum sequence over $G\}$ the *Davenport constant* of G.

For any integer-valued function $f : A \to \mathbb{Z}$ defined on a finite set A, we denote by $\min(f) = \min\{f(a) \mid a \in A\}$ and by $\max(f) = \max\{f(a) \mid a \in A\}$.

Next we list some necessary lemmas.

Lemma 2.1. *(*[7, Lemma 4.4]*) Let G be a finite abelian group and $X \subseteq G \setminus \{0\}$ be a generating set for G. Suppose that $f : G \to \mathbb{Z}$ is a function on G. Then*

$$\sum_{x\in X,\, g\in G} \max\{f(g+x) - f(g), 0\} \geq (\max(f) - \min(f))|X|.$$

Lemma 2.2. *Let G be a finite abelian group, $H \subseteq G$ a subgroup, $S \subseteq G$ a subset, and let $f : G/H \to \mathbb{Z}$ be defined by $f(a+H) = |(a+H) \bigcap S|$ for all $a \in G$.*

Suppose that $X \subseteq G\setminus\{0\}$ is a generating set for G and satisfies $|(S+x)\setminus S| \leq 7$ for all $x \in X$. Then

$$\min(f) \geq \max(f) - 7.$$

In particular, if there exists an element $b \in G$ such that $f(b+H) \geq 8$, then $f(a+H) \geq 1$ for all $a \in G$.

Proof. Obviously, the assertion holds for $H = \{0\}$ and for $H = G$. Suppose that $\{0\} \subsetneq H \subsetneq G$. Let $A \subseteq G$ be such that $G = \bigcup_{a\in A}(a+H)$ and $|A| = |G/H|$. Since $\{x + H \mid x \in X\}$ is a generating set for G/H, choose $X' \subseteq X$ such that $\{x + H \mid x \in X'\}$ is a generating set for G/H and $|\{x + H \mid x \in X'\}| = |X'|$.

By $|(S+x) \setminus S| \leq 7$ for all $x \in X$,

$$\begin{aligned} 7|X'| &\geq \sum_{x\in X'} |(S+x)\setminus S| \\ &= \sum_{x\in X'}\sum_{a\in A} \big|((S+x)\cap(a+H)) \setminus (S\cap(a+H))\big| \\ &\geq \sum_{x\in X'}\sum_{a\in A} \max\{f(a-x+H) - f(a+H), 0\} \\ &= \sum_{x\in -X'}\sum_{a\in A} \max\{f(a+H+x+H) - f(a+H), 0\}. \end{aligned}$$

Since $\{x + H \mid x \in X'\}$ is a generating set for G/H, we get $\{x + H \mid x \in -X'\}$ is also a generating set for G/H. Therefore, by Lemma 2.1,

$$7|X'| \geq \sum_{x\in -X'}\sum_{a\in A} \max\{f(a+H+x+H) - f(a+H), 0\} \geq (\max(f) - \min(f))|X'|.$$

By $|X'| \neq 0$, we obtain

$$\min(f) \geq \max(f) - 7.$$

In particular, if $f(b+H) \geq 8$ for some $b \in G$, then for all $a \in G$, $f(a+H) \geq \min(f) \geq \max(f) - 7 \geq f(b+H) - 7 \geq 1$. □

We also need the following simple and well-known result.

Lemma 2.3. *Let G be a finite abelian group and S be a zero-sum free sequence over G. Then*

(1) $|\Sigma(S)| \geq |S|$,
(2) $\mathsf{D}(G) \leq |G|$.

Proof. (1) Suppose $S = g_1 \cdot \ldots \cdot g_l$. Then $g_1, g_1 + g_2, \ldots, g_1 + \ldots + g_l$ are all distinct. It follows that $|\Sigma(S)| \geq l = |S|$.
(2) Assume to the contrary that X is a zero-sum free sequence over G with length $|G|$. Thus by (1), $|\Sigma(X)| \geq |G|$ which implies that $0 \in \Sigma(X)$, a contradiction.

□

Lemma 2.4. *Let G be a finite abelian group and $X = X_1X_2$ be a zero-sum free sequence over G, then*

(1) $|\Sigma(X)| \geq |\Sigma(X_1)| + |\Sigma(X_2)|$.
(2) Let $H = \langle \mathrm{supp}(X_1) \rangle$ and let $\varphi : G \to G/H$ be the canonical epimorphism. If $\varphi(X_2)$ is a zero-sum free sequence over G/H, then

$$|\Sigma(X)| \geq (|\Sigma(X_1)| + 1)(|X_2| + 1) - 1.$$

Proof. (1) This follows by [3, Theorem 5.3.1].
(2) Since $\varphi(X_2)$ is a zero-sum free sequence over G/H, we get $H \cap \Sigma(X_2) = \emptyset$ and $|\Sigma(\varphi(X_2))| \geq |X_2|$ by Lemma 2.3.1. Thus for any $a \in \Sigma(X_2)$,

$$|\Sigma(X_0) \cap (a+H)| \geq |(\Sigma(X_1) + a) \cup \{a\}| = |\Sigma(X_1)| + 1.$$

Therefore

$$\begin{aligned} |\Sigma(X_0)| &\geq |\Sigma(X_0) \cap H| + \sum_{a+H \in \Sigma(\varphi(X_2))} |\Sigma(X_0) \cap (a+H)| \\ &\geq |\Sigma(X_1)| + (|\Sigma(X_1)| + 1)|\Sigma(\varphi(X_2))| \\ &\geq (|\Sigma(X_1)| + 1)(|X_2| + 1) - 1. \end{aligned}$$

□

3. The proof of Theorem 1.1

For the simplicity of formulations, we define *C-sequences* and *C-groups*. To begin with, a sequence X over a finite abelian group G is called a *C-sequence* if the following three conditions hold:

(i) $\langle \mathrm{supp}(X)\rangle = G$,
(ii) X is zero-sum free,
(iii) $|\sum(X)| \le 8|X| - 18$.

Furthermore, a finite abelian group G is called a *C-group* if the following three conditions hold:

(i) $\mathsf{r}(G) = 4$,
(ii) the smallest prime p dividing $|G|$ satisfies $p \ge 13$,
(iii) there exists a *C-sequence* over G.

Proof of Theorem 1.1. If Theorem 1.1 does not hold, then there exists a *C-group*. Let G_0 be the *C-group* with minimal order and let X_0 be a *C-sequence* over G_0 with minimal length.

We proceed by the following four claims:

Claim A. Let X be a zero-sum free sequence over G_0 and $H = \langle \mathrm{supp}(X)\rangle$ with $r = \mathsf{r}(H)$. If $|H| < |G_0|$ or $|X| < |X_0|$, then

$$|\Sigma(X)| \ge 2^{r-1}(|X| - r + 2) - 1.$$

Proof. By Theorem A and these hypothesis about G_0 and X_0, it follows directly! □

Claim B. (1) Let H be a subgroup of G_0. Then for any $a \in G_0$,

$$|\Sigma(X_0) \cap (a + H)| \ge \max_{g \in G_0} |\Sigma(X_0) \cap (g + H)| - 7 \ge |\Sigma(X_0) \cap H| - 7.$$

(2) Suppose that X_0 has a factorization $X_0 = X_1X_2$ such that $H = \langle \mathrm{supp}(X_1)\rangle$ is a proper subgroup of G. If $|\Sigma(X_1)| \ge 7$, then

$$|\Sigma(X_0)| \ge (\Sigma(X_1) + 1)|G/H| - 1.$$

Proof. (1) For any $x \,|\, X_0$, denote $H_x = \langle \mathrm{supp}(X_0x^{-1})\rangle$. Then $\mathsf{r}(H_x) \ge \mathsf{r}(G_0) - 1 = 3$. By Claim A and $|X_0x^{-1}| < |X_0|$, we get

$$|\Sigma(X_0x^{-1})| \ge \min\{4(|X_0| - 1 - 3 + 2) - 1,\ 8(|X_0| - 1 - 4 + 2) - 1\} = 4|X_0| - 9.$$

If $H_x \ne G_0$, then $x \notin H_x$. Thus $|\Sigma(X_0)| \ge 2|\Sigma(X_0x^{-1})| + 1 \ge 8|X_0| - 17$ by Lemma 2.4.2, a contradiction to that X_0 is a C-sequence.

Therefore $H_x = G_0$ and $\mathsf{r}(H_x) = 4$. By Claim A and $|X_0x^{-1}| < |X_0|$,

$$|\Sigma(X_0x^{-1})| \ge 8(|X_0| - 1) - 17 = 8|X_0| - 25.$$

Let $S = \Sigma(X_0)$. Then $|S| \le 8|X_0| - 18$ and for all $x \,|\, X_0$,

$$\begin{aligned}|(S + x) \setminus S| &= |S \setminus (S - x)| \le |S \setminus \Sigma(X_0x^{-1})| \\ &\le |S| - |\Sigma(X_0x^{-1})| \le (8|X_0| - 18) - (8|X_0| - 25) \le 7.\end{aligned}$$

By $\langle \mathrm{supp}(X_0)\rangle = G_0$ and Lemma 2.2, for any $a \in G$,

$$|\Sigma(X_0) \cap (a + H)| \geq \max_{g \in G_0} |\Sigma(X_0) \cap (g + H)| - 7 \geq |\Sigma(X_0) \cap H| - 7.$$

(2) Since H is a proper subgroup of G_0, there exists $x \mid X_2$ such that $x \notin H$. Then

$$|\Sigma(X_0) \cap (x + H)| \geq |(\Sigma(X_1) + x) \cup \{x\}| \geq |\Sigma(X_1)| + 1 \geq 8.$$

For any $a \in G_0 \backslash H$, we get that $|\Sigma(X_0) \cap (a+H)| \geq |\Sigma(X_0) \cap (x+H)| - 7 \geq 1$ by (1) which implies that $\Sigma(X_2) \cap (a + H) \neq \emptyset$.

Choose $b \in \Sigma(X_2) \cap (a+H)$. Then $|\Sigma(X_0) \cap (a+H)| \geq |(\Sigma(X_1)+b) \cup \{b\}| = |\Sigma(X_1)| + 1$ for all $a \in G_0 \setminus H$.

Therefore

$$\begin{aligned} |\Sigma(X_0)| &\geq |\Sigma(X_1)| + (|\Sigma(X_1)| + 1)(|G/H| - 1) \\ &\geq (|\Sigma(X_1)| + 1)(|G/H|) - 1. \end{aligned}$$

□

Claim C. Let X be a subsequence of X_0. If $H = \langle \mathrm{supp}(X)\rangle$ is a proper subgroup of G_0, then $\mathsf{r}(H) \leq 3$.

Proof. Assume to the contrary that $\mathsf{r}(H) = 4$. Then $|X| \geq 4$.

Let $\varphi : G_0 \to G_0/H$ denote the canonical epimorphism from G_0 to G_0/H with $\ker(\varphi) = H$. Then $\varphi(X_0)$ is a sequence over G_0/H. We can get a factorization of X_0,

$$X_0 = X \cdot X_1 \cdot \ldots \cdot X_\alpha \cdot X',$$

satisfying that for $1 \leq i \leq \alpha$, $\varphi(X_i)$ is a minimal zero-sum sequence over G_0/H and $\varphi(X')$ is a zero-sum free sequence over G_0/H. Thus $|\Sigma(\varphi(X'))| \geq |X'|$ and $|X_0| \leq |X| + \alpha \mathsf{D}(G/H) + |X'| \leq |X| + \alpha|G/H| + |X'|$ by Lemma 2.3.

Let $Y = X \cdot \sigma(X_1) \cdot \ldots \cdot \sigma(X_\alpha)$. Then Y is a zero-sum free sequence over H. By $H < G_0$ and Claim A, we have

$$|\Sigma(X_0) \cap H| \geq |\Sigma(Y) \cap H| \geq 8|Y| - 17.$$

For any $a \in \Sigma(X')$, we get $a \notin H$ and

$$|\Sigma(X_0) \cap (a + H)| \geq |\Sigma(Y \cdot a) \cap (a + H)| \geq |\Sigma(Y) \cap H| + 1 \geq 8|Y| - 16.$$

Let $A' \subseteq \Sigma(X')$ satisfy $\{a + H \mid a \in \Sigma(X')\} = \{a + H \mid a \in A'\}$ and $|A'| = |\varphi(\Sigma(X'))|$.

Let $A \subseteq G_0$ be a subset with $A \supseteq A'$ such that $G_0 = \cup_{a \in A}(a + H)$ and $|A| = |G_0/H|$. Then for any $b \in A \setminus (A' \cup H)$,

$$|\Sigma(X_0) \cap (b + H)| \geq |\Sigma(X_0) \cap H| - 7 \geq 8|Y| - 24,$$

by Claim B.1.

Therefore,

$$\begin{aligned}|\Sigma(X_0)| &= \sum_{a\in A} |\Sigma(X_0)\cap(a+H)|\\ &\geq 8|Y|-17+(8|Y|-16)|\Sigma(\varphi(X'))|+(8|Y|-24)(|G/H|-1-|\Sigma(\varphi(X'))|)\\ &\geq (8|Y|-24)|G/H|+8|\Sigma(\varphi(X'))|+7\\ &\geq 8(|X|-3)(|G|/|H|-1)+8(|X|+\alpha|G|/|H|+|X'|)-17\\ &\geq 8|X_0|-17.\end{aligned}$$

A contradiction. □

Claim D. Let Y be a subsequence of X_0 with length 4. Then $\langle \mathrm{supp}(Y)\rangle = G_0$.

Proof. Let X be the longest subsequence of X_0 such that $\langle \mathrm{supp}(X)\rangle \neq G_0$. Denote $H = \langle \mathrm{supp}(X)\rangle$. Then $\mathsf{r}(H) = 3$ by Claim C and $|G_0/H| \geq 13$ since G_0 is a *C-group*. Let $\varphi : G_0 \to G_0/H$ denote the canonical epimorphism.

We only need to prove that $|X| \leq 3$. Assume to the contrary that $|X| \geq 4$. We distinguish three cases to finish the proof.

CASE 1: $|X_0| \leq \frac{(|X|-1)|G_0/H|+4}{2}$.

By $H < G_0$ and Claim A, $|\Sigma(X)| \geq 4(|X|-1)-1 \geq 11$. Then by Claim B.2,

$$|\Sigma(X_0)| \geq (|\Sigma(X)|+1)|G_0/H|-1 \geq 4(|X|-1)|G_0/H|-1,$$

which implies that $|\Sigma(X_0)| \geq 8|X_0|-17$ by $|X_0| \leq \frac{(|X|-1)|G_0/H|+4}{2}$, a contradiction.

CASE 2: There exists no zero-sum free subsequence of $\varphi(X_0X^{-1})$ with length 6.

Since $\varphi(X_0)$ is a sequence over G_0/H, we can get a factorization of X_0,

$$X_0 = X\cdot X_1\cdot\ldots\cdot X_\alpha\cdot X',$$

satisfying that for $1 \leq i \leq \alpha$, $\varphi(X_i)$ is a minimal zero-sum sequence over G_0/H and $\varphi(X')$ is a zero-sum free sequence over G_0/H. Thus $|X_0| = |X|+|X_1|+\ldots+|X_\alpha|+|X'| \leq |X|+|X'|+6\alpha$ and $|\Sigma(\varphi(X'))| \geq |X'|$ by Lemma 2.3.

Let $Y = X\cdot\sigma(X_1)\cdot\ldots\cdot\sigma(X_\alpha)$. Then Y is a zero-sum free sequence over H. By Claim A and $H < G_0$, we have

$$|\Sigma(X_0)\cap H| \geq |\Sigma(Y)\cap H| \geq 4|Y|-5.$$

For any $a \in \Sigma(X')$, we obtain $a \notin H$ and

$$|\Sigma(X_0)\cap(a+H)| \geq |\Sigma(Y\cdot a)\cap(a+H)| \geq |\Sigma(Y)\cap H|+1 \geq 4|Y|-4.$$

Let $A' \subseteq \Sigma(X')$ satisfy $\{a+H \mid a\in\Sigma(X')\} = \{a+H \mid a \in A'\}$ and $|A'| = |\varphi(\Sigma(X'))|$.

Let $A \subseteq G_0$ be a subset with $A \supseteq A'$ such that $G_0 = \cup_{a\in A}(a+H)$ and $|A| = |G_0/H|$. Then for any $b \in A\setminus(\Sigma(X')\cup H)$,

$$|\Sigma(X_0)\cap(b+H)| \geq |\Sigma(X_0)\cap H|-7 \geq 4|Y|-12,$$

by Claim B.1.

Therefore,

$$\begin{aligned}|\Sigma(X_0)| &= \sum_{a\in A} |\Sigma(X_0)\cap(a+H)|\\ &\ge 4|Y|-5+(4|Y|-4)|\Sigma(\varphi(X'))|+(4|Y|-12)(|G/H|-1-|\Sigma(\varphi(X'))|)\\ &\ge (4|Y|-12)|G/H|+8|\Sigma(\varphi(X'))|+7\\ &\ge (4|X|+4\alpha-12)|G/H|+8|X'|+7.\end{aligned}$$

Since $|X|\ge 4$, $|G/H|\ge 13$, and $|X_0|\le |X|+|X'|+6\alpha$, we have that $|\Sigma(X_0)|\ge 8|X_0|-17$, a contradiction.

CASE 3: $|X_0| > \frac{(|X|-1)|G/H|+4}{2}$ and there exists a subsequence X_1 of X_0X^{-1} such that $\varphi(X_1)$ is a zero-sum free subsequence over G_0/H with length 6.

Since $|X_0| > \frac{(|X|-1)|G/H|+4}{2}$, we obtain $|X_0|-2|X| > \frac{|X|(|G/H|-4)-|G/H|+4}{2}\ge 7$.

Denote $X_2 = X_0(XX_1)^{-1}$. Then $|X_2| = |X_0|-|XX_1| > |X|+1$. Thus $\langle X_2\rangle = G_0$ since X is the longest subsequence of X_0 such that $\langle \mathrm{supp}(X)\rangle \ne G_0$. Then $|\Sigma(X_2)|\ge 8|X_2|-17$ by $|X_2|<|X_0|$ and Claim A.

By $H<G_0$ and Claim A, $|\Sigma(X)|\ge 4(|X|-1)-1$. It follows that $|\Sigma(XX_1)|\ge 4(|X|-1)(|\Sigma(\varphi(X_1))|+1)-1\ge 8|XX_1|$ by Lemma 2.4.2, $|X|\ge 4$ and $|X_1|=6$.

Therefore by Lemma 2.4.1, $|\Sigma(X_0)|\ge|\Sigma(XX_1)|+|\Sigma(X_2)|\ge 8|XX_1|+8|X_2|-17 = 8|X_0|-17$, a contradiction. □

Now we finish the proof of Theorem 1.1 by distinguishing the following two cases.

Suppose that $|X_0|\ge 13$. Denote $X_0 = x_1\cdot\ldots\cdot x_n$. Then by Claim D, any four elements of X_0 are independent which implies that $x_i, x_j+x_k, 1\le i\le n, 1\le j<k\le n$ are all different elements in G_0. Therefore,

$$|\Sigma(X_0)|\ge n+\frac{n(n-1)}{2}\ge 8|X_0|-17\,,$$

a contradiction.

Suppose that $|X_0|\le 12$. Let X be a subsequence of X_0 with length 3 and $H=\langle\mathrm{supp}(X)\rangle$ is a proper subgroup of G_0. Then by Claim D, the three elements of X must be independent which implies that $|\Sigma(X)|=7$. It follows by Claim B.2 that

$$|\Sigma(X_0)|\ge(|\Sigma(X)|+1)|G_0/H|-1\ge 8\cdot 13-1\ge 8|X_0|-17\,,$$

a contradiction.

□

Acknowledgements. The authors are grateful to the referee for helpful suggestions and comments. This research was supported by NSFC (grant no. 11371184, 11426128), NSF of Henan Province (grant no. 142300410304), the Education

Department of Henan Province (grant no. 2009A110012), NSF of Luoyang Normal University (grant no. 10001199), and the Austrian Science Fund FWF(project no. M1641-N26).

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