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Two zero-sum invariants on finite abelian groups

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ABSTRACT

Let G be an additive finite abelian group with exponent $\exp(G)$. Let $\mathsf{s}(G)$ (resp. $\eta(G)$) be the smallest integer t such that every sequence of t elements (repetition allowed) from G contains a zero-sum subsequence T of length $|T| = \exp(G)$ (resp. $|T| \in [1, \exp(G)]$). Let H be an arbitrary finite abelian group with $\exp(H) = m$. In this paper, we show that $\mathsf{s}(C_{mn} \oplus H) = \eta(C_{mn} \oplus H) + mn - 1$ holds for all $n \geq \max\{m|H| + 1, 4|H| + 2m\}$.

In memory of Y.O. Hamidoune

1. Introduction

Let G be an additive finite abelian group with exponent $\exp(G) = m$. Let $\mathsf{D}(G)$ denote the Davenport constant of G, which is defined as the smallest integer t such that every sequence S over G of length $|S| \geq t$ contains a nonempty zero-sum subsequence. For every positive integer k, let $\mathsf{s}_{km}(G)$ denote the smallest integer t such that every sequence S over G of length $|S| \geq t$ contains a zero-sum subsequence of length km. For $k = 1$, we abbreviate $\mathsf{s}_m(G)$ to $\mathsf{s}(G)$ which is called the Erdős–Ginzburg–Ziv constant of G. The famous Erdős–Ginzburg–Ziv theorem asserts that $\mathsf{s}_{|G|}(G) \leq 2|G| - 1$ and equality holds for cyclic groups. Let $\eta(G)$ be the smallest integer t such that every sequence S over G of length $|S| \geq t$ contains a zero-sum subsequence of length in $[1, m]$. The constants $\mathsf{s}(G)$ and $\eta(G)$ are classical invariants in zero-sum theory and have received a lot of attention (For example, see [2–4,6–11,15–17,19,20,25–28,30–33]). For some recent progress on them we refer to [20]. Our main motivation is the following conjecture suggested by the second author [14] in 2003.

Conjecture 1.1. $\mathsf{s}(G) = \eta(G) + \exp(G) - 1$ *holds for all finite abelian groups* G.

This conjecture holds true for all G when $\mathsf{s}(G)$ has been determined. For the case that $\mathsf{s}(G)$ is unknown, this conjecture has been confirmed [14] only for $\exp(G) \in \{3, 4\}$. In this paper we shall prove

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Theorem 1.2. *Let H be an arbitrary finite abelian group with $\exp(H) = m \geq 2$, and let $G = C_{mn} \oplus H$. If $n \geq \max\{m|H| + 1, 4|H| + 2m\}$ then $\mathsf{s}(G) = \eta(G) + \exp(G) - 1$.*

Theorem 1.3. *Let G be a finite abelian group, and let S be a sequence over G of length $|S| = \mathsf{s}(G) - 1$ such that $0 \notin \sum_{\exp(G)}(S)$. If $\max_{g\in G}\{v_g(S)\} \geq \lfloor \frac{\exp(G)-1}{2} \rfloor$ then $\mathsf{s}(G) = \eta(G) + \exp(G) - 1$.*

Corollary 1.4. *Let m be a positive integer, and let H be a finite abelian group with $\exp(H)|m$ and $\mathsf{D}(H) = m$. Suppose that $\mathsf{D}(C_m \oplus C_m \oplus H) = 2m + \mathsf{D}(H) - 2$. If $n \geq \max\{m|H| + 1, 4|H| + 2m\}$ then $\mathsf{s}(C_{mn} \oplus H) = \eta(C_{mn} \oplus H) + mn - 1 = (2n + 1)m + \mathsf{D}(H) - 3$.*

2. Preliminaries

Our notation and terminology are consistent with [15,21]. We briefly gather some key notions and fix the notations concerning sequences over finite abelian groups. Let $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any two integers $a, b \in \mathbb{N}$, we set $[a, b] = \{x \in \mathbb{N} : a \leq x \leq b\}$. Throughout this paper, all abelian groups will be written additively, and for $n, r \in \mathbb{N}$, we denote by C_n the cyclic group of order n, and denote by C_n^r the direct sum of r copies of C_n.

Let G be a finite abelian group and $\exp(G)$ its exponent. A sequence S over G will be written in the form

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g\in G} g^{\mathsf{v}_g(S)}, \quad \text{with } \mathsf{v}_g(S) \in \mathbb{N}_0 \text{ for all } g \in G,$$

and we call

$$|S| = \ell \in \mathbb{N}_0 \text{ the } \textit{length} \text{ and } \sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g\in G} \mathsf{v}_g(S) g \in G \text{ the } \textit{sum} \text{ of } S.$$

Let $\mathrm{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$. For every $r \in [1, \ell]$ define

$$\sum_r(S) = \{\sigma(T) : T \mid S, |T| = r\}$$

where $T \mid S$ means T is a subsequence of S.

The sequence S is called

- a *zero-sum sequence* if $\sigma(S) = 0$.
- a *short zero-sum sequence* over G if it is a zero-sum sequence of length $|S| \in [1, \exp(G)]$.

For every element $g \in G$, we set $g + S = (g + g_1) \cdot \ldots \cdot (g + g_l)$. If $\varphi: G \to H$ is a group homomorphism, then $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$ is a zero-sum sequence if and only if $\sigma(S) \in \ker(\varphi)$.

Lemma 2.1 (*[12,19], [21, Theorem 5.7.4]*)**.** $\mathsf{s}(G) \leq |G| + \exp(G) - 1$.

Lemma 2.2. $\mathsf{s}(G) \geq \eta(G) + \exp(G) - 1$.

Proof. Let S be a sequence over G of length $|S| = \eta(G) - 1$ such that S contains no short zero-sum subsequence. Then, the sequence $0^{\exp(G)-1}S$ contains no zero-sum subsequence of length $\exp(G)$. Therefore, $\mathsf{s}(G) \geq 1 + |0^{\exp(G)-1}S| = \eta(G) + \exp(G) - 1$. □

Lemma 2.3. *Let n, k be two positive integers with $2 \leq k < \frac{n}{2}$, and let S be a sequence over C_n of length $|S| = 2n - k$. Suppose that S contains no zero-sum subsequence of length n. Then,*

(i) *there exist two distinct elements $a, b \in C_n$ such that*

$$v_a(S) + v_b(S) \geq 2n - 2k + 2. \tag{1}$$

(ii) *Let $a_1, b_1, a_2, b_2 \in C_n$ with $a_1 \neq b_1$ and $a_2 \neq b_2$. If $v_{a_1}(S) + v_{b_1}(S) + v_{a_2}(S) + v_{b_2}(S) \geq 3n - 2$ then $\{a_1, b_1\} = \{a_2, b_2\}$.*

(iii) *If $k < \frac{n+7}{4}$ then the pair $\{a, b\}$ in the inequality (1) is uniquely determined by S.*

Proof. (i) The existence of the pair $\{a, b\}$ satisfying inequality (1) follows from [34, Corollary 7].

(ii) Assume to the contrary that $\{a_1, b_1\} \neq \{a_2, b_2\}$. Without loss of generality we assume that $a_2 \notin \{a_1, b_1\}$. It follows that

$$v_{a_1}(S) + v_{b_1}(S) + v_{a_2}(S) \leq |S|.$$

Since S contains no zero-sum subsequence of length n, $v_{b_2}(S) \leq n-1$ and $|S| \leq 2n-2 = \mathsf{s}(C_n) - 1$ by the Erdős–Ginzburg–Ziv theorem. Therefore, $v_{a_2}(S) \geq v_{a_2}(S) + v_{b_2}(S) - n + 1$. Hence, $2n-2 \geq |S| \geq v_{a_1}(S) + v_{b_1}(S) + v_{a_2}(S) \geq v_{a_1}(S) + v_{b_1}(S) + v_{a_2}(S) + v_{b_2}(S) - n + 1$. This gives that $v_{a_1}(S) + v_{b_1}(S) + v_{a_2}(S) + v_{b_2}(S) \leq 3n-3$, a contradiction.

(iii) Now suppose that $k < \frac{n+7}{4}$. Note that $2(2n-2k+2) = 4n-4k+4 > 3n-3$. Now (iii) follows from (ii). □

The following three easy lemmas will be used in the proof of Corollary 1.4.

Lemma 2.4 ([7, Lemma 3.2]). *Let $n \in \mathbb{N}$, and let H be a finite abelian group with $\exp(H) | n$. Let $G = C_n \oplus H$. Then, $\eta(G) \geq n + 2(\mathsf{D}(H) - 1)$.*

Lemma 2.5 ([21, Proposition 5.7.11]). *Let G be a finite abelian group, and let H be a subgroup of G with $\exp(G) = \exp(H)\exp(G/H)$. Then, $\eta(G) \leq \exp(G/H)(\eta(H) - 1) + \eta(G/H)$.*

Lemma 2.6. *Let H and K be two finite abelian groups, and let $G = H \oplus K$. Then, $\mathsf{D}(G) \geq \mathsf{D}(H) + \mathsf{D}(K) - 1$.*

We also need the following technical result for the proof of Corollary 1.4.

Lemma 2.7. *Let m be a positive integer, and let H be a finite abelian group with $m \geq \mathsf{D}(H)$. Suppose that $\mathsf{D}(C_m \oplus C_m \oplus H) = 2m + \mathsf{D}(H) - 2$. Then, $\eta(C_m \oplus H) \leq 2m + \mathsf{D}(H) - 2$.*

Proof. From Lemma 2.6 and the well known fact that $\mathsf{D}(C_m) = m$ we infer that, $2m + \mathsf{D}(H) - 2 = \mathsf{D}(C_m \oplus C_m \oplus H) \geq m + \mathsf{D}(C_m \oplus H) - 1 \geq 2m + \mathsf{D}(H) - 2$. This forces that

$$\mathsf{D}(C_m \oplus H) = m + \mathsf{D}(H) - 1 < 2m.$$

Let S be a sequence over $C_m \oplus H$ of length $|S| = 2m + \mathsf{D}(H) - 2$. We need to show that S contains a short zero-sum subsequence over $C_m \oplus H$. Obviously, $\exp(C_m \oplus H) \geq m$. So, it suffices to prove that S contains a zero-sum subsequence of length in $[1, m]$. Let $r = |S|$ and let $S = g_1 \cdot \ldots \cdot g_r$. Let $G = C_m \oplus C_m \oplus H = \langle e_1 \rangle \oplus C_m \oplus H$ with $\langle e_1 \rangle = C_m$. Consider the sequence

$$W = (e_1, g_1) \cdot \ldots \cdot (e_1, g_r)$$

over G.

Since $r = 2m + \mathsf{D}(H) - 2 = \mathsf{D}(G)$, W contains a nonempty zero-sum subsequence T. By the construction of W we see that $|T| \in \{m, 2m\}$. Suppose $T = \prod_{i \in I}(e_1, g_i)$ with $|I| \in \{m, 2m\}$. Then, $S_1 = \prod_{i \in I} g_i$ is a zero-sum subsequence of S of length $|S_1| \in \{m, 2m\}$. If $|S_1| = m$ then we are done. Otherwise, $|S_1| = 2m > \mathsf{D}(C_m \oplus H)$. It follows that S_1 contains a zero-sum subsequence S_2 (say) such that $1 \leq |S_2| \leq \mathsf{D}(C_m \oplus H) < |S_1|$. Therefore, either S_2 or $S_1 S_2^{-1}$ is a short zero-sum sequence over $C_m \oplus H$. □

3. Proofs of Theorems 1.2, 1.3 and Corollary 1.4

Proof of Theorem 1.2. By Lemma 2.2 we only need to show $\eta(G) - 1 \geq \mathsf{s}(G) - \exp(G)$. Let $\varphi : C_{mn} \oplus H \to C_m \oplus H$ be the natural homomorphism with $\ker(\varphi) = C_n$ (up to isomorphism). By the definition of $\mathsf{s}(G)$, there exists a sequence S over G such that $|S| = \mathsf{s}(G) - 1$ and $0 \notin \sum_{mn}(S)$.

Apply $\mathsf{s}(\varphi(C_{mn} \oplus H)) = \mathsf{s}(C_m \oplus H)$ to $\varphi(S)$ repeatedly, we can get a decomposition $S = S_1 \cdot \ldots \cdot S_r \cdot S'$ with

$$|S_i| = m, \sigma(S_i) \in \ker(\varphi) \quad \text{for every } i \in [1, r] \tag{2}$$

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and $\mathsf{s}(C_m \oplus H) - m \le |S'| \le \mathsf{s}(C_m \oplus H) - 1$. Therefore,

$$r = \left\lceil \frac{|S| - \mathsf{s}(C_m \oplus H) + 1}{m} \right\rceil. \tag{3}$$

Let $U = \sigma(S_1)\sigma(S_2)\cdots\sigma(S_r)$. It follows from $0 \notin \sum_{mn}(S)$ that $0 \notin \sum_n(U)$. Since $\mathsf{s}(G) \ge \mathsf{s}(C_{mn}) = 2mn - 1$ and $\mathsf{s}(C_m \oplus H) \le m \cdot |H| + m - 1$ by Lemma 2.1, we infer that

$$|U| = r \ge \frac{\mathsf{s}(G) - \mathsf{s}(C_m \oplus H)}{m} \ge \frac{2mn - 1 - (m \cdot |H| + m - 1)}{m} = 2n - |H| - 1. \tag{4}$$

Let $k = 2n - r$. Since $0 \notin \sum_n(U)$, $r = |U| \le 2n - 2$ by the Erdős–Ginzburg–Ziv theorem. It follows that $k \ge 2$. By (4) and the hypothesis that $n \ge \max\{m|H| + 1, 4|H| + 2m\} > 4|H| - 2$ we get, $k < \frac{n+7}{4}$. It follows from Lemma 2.3 that there exists a unique pair $\{a, b\}$ such that

$$v_a(U) + v_b(U) \ge 2n - 2k + 2.$$

Denote by Ω the set consisting of all decompositions of S satisfying (2) and (3). Choose a decomposition $S = S_1 \cdot S_2 \cdot \ldots \cdot S_r \cdot S' \in \Omega$ such that $v_a(U) + v_b(U)$ attains the minimal value. Let $\ell = v_a(U) + v_b(U)$. By renumbering if necessary we assume that $\sigma(S_i) \in \{a, b\}$ for all $i \in [1, \ell]$. Let

$$W = \prod_{i=1}^{\ell} S_i.$$

From $k < \frac{n+7}{4}$ and $n \ge 4|H| + 2m > 2m$ we derive that $\ell \ge 2n - 2k + 2 > m$.

Claim 1. *Let W_0 be a subsequence of W of length $|W_0| = m$. If $\sigma(W_0) \in \ker(\varphi)$ then $\sigma(W_0) \in \{a, b\}$.*

Assume to the contrary that $\sigma(W_0) \notin \{a, b\}$. Since $|W_0| = m$, by renumbering we may assume that $W_0 | S_1 \cdot S_2 \cdot \ldots \cdot S_m$. Then S has a decomposition

$$S = S_{m+1} \cdot S_{m+2} \cdot \ldots \cdot S_r \cdot W_0 \cdot S_2' \cdot S_3' \cdot \ldots \cdot S_m' \cdot S'' \in \Omega$$

where $|S_i'| = m$ and $\sigma(S_i') \in \ker(\varphi)$ for every $i \in [2, m]$.

Let $U_1 = \sigma(S_{m+1}) \cdot \sigma(S_{m+2}) \cdot \ldots \cdot \sigma(S_r) \cdot \sigma(W_0) \cdot \sigma(S_2') \cdot \ldots \cdot \sigma(S_m')$. By Lemma 2.3(i) and (iii), there is a unique pair $\{a_1, b_1\}$ such that $v_{a_1}(U_1) + v_{b_1}(U_1) \ge 2n - 2k + 2$. Note that $v_a(U_1) + v_b(U_1) \ge v_a(U) + v_b(U) - m \ge 2n - 2k + 2 - m$. Since $n \ge 4|H| + 2m$ and (4), we infer that $v_{a_1}(U_1) + v_{b_1}(U_1) + v_a(U_1) + v_b(U_1) \ge 2(2n - 2k + 2) - m \ge 3n - 2$. So, $\{a, b\} = \{a_1, b_1\}$ by Lemma 2.3. But $v_a(U_1) + v_b(U_1) < v_a(U) + v_b(U)$, a contradiction to the minimality of U. This proves Claim 1.

Since $n \ge m|H| + 1$ and (4), $|\varphi(W)| = (v_a(U) + v_b(U)) \cdot m \ge (2n - 2|H|)m > (2m - 2) \cdot |C_m \oplus H|$. Therefore, there exist an element $h_0 \in C_m \oplus H$ and a subsequence W_1 of W such that $\varphi(W_1) = h_0^{2m-1}$. Now we have the following.

Claim 2. *There exists an element $g_0 \in G$ such that $g_0^m | W_1$.*

Suppose that there are three distinct elements $g_0, g_1, g_2 \in \mathrm{supp}(W_1)$. Let W_2 be a subsequence of $W_1(g_0g_1g_2)^{-1}$ of length $|W_2| = m - 2$. Then, $W_2g_0g_1$, $W_2g_0g_2$ and $W_2g_1g_2$ are three subsequences of W_1 each having sum in $\ker(\varphi) = C_n$. But the sums $\sigma(W_2g_0g_1)$, $\sigma(W_2g_0g_2)$, $\sigma(W_2g_1g_2)$ are distinct, a contradiction to Claim 1. Therefore, $|\mathrm{supp}(W_1)| \le 2$. Now Claim 2 follows from $|W_1| = 2m - 1$.

Without loss of generality, we assume that $\sigma(g_0^m) = mg_0 = a$ and $g_0^m | S_1 \cdot S_2 \cdot \ldots \cdot S_m$. Then S has the following decomposition:

$$S = S_{m+1} \cdot S_{m+2} \cdot \ldots \cdot S_r \cdot g_0^m \cdot W_2' \cdot W_3' \cdot \ldots \cdot W_m' \cdot W' \in \Omega$$

where $|W_i'| = m$ and $\sigma(W_i') \in \ker(\varphi)$ for every $i \in [2, m]$.

Let $U_2 = \sigma(S_{m+1}) \cdot \sigma(S_{m+2}) \cdot \ldots \cdot \sigma(S_r) \cdot a \cdot \sigma(W_2') \cdot \ldots \cdot \sigma(W_m')$. Since $n \ge 4|H| + 2m$, from (4) we infer that $v_a(U_2) + v_b(U_2) \ge v_a(U) + v_b(U) - (m - 1) \ge \frac{3}{2}n$. Therefore,

$$v_a(U_2) \ge \frac{3}{2}n - v_b(U_2) \ge \frac{3}{2}n - (n - 1) > \frac{n}{2}.$$

Let

$$T = -g_0 + S = (-g_0 + S_{m+1}) \cdot (-g_0 + S_{m+2}) \cdot \ldots \cdot (-g_0 + S_r) \cdot (-g_0 + g_0^m) \cdot (-g_0 + W_2') \cdot (-g_0 + W_3') \cdot \ldots \cdot (-g_0 + W_m') \cdot (-g_0 + W') = T_1 \cdot T_2 \cdot \ldots \cdot T_r \cdot T'$$

where $T' = -g_0 + W'$, and $T_1, \ldots, T_r$ is a permutation of $(-g_0 + S_{m+1}), (-g_0 + S_{m+2}), \ldots, (-g_0 + S_r), (-g_0 + g_0^m), (-g_0 + W_2'), (-g_0 + W_3'), \ldots, (-g_0 + W_m')$ such that $T_1 = -g_0 + g_0^m = 0^m$ and $\sigma(T_i) = 0$ for every $i \in [2, \lceil \frac{n}{2} \rceil]$. It follows from $0 \notin \sum_{mn}(S)$ that

$$0 \notin \sum_{mn}(T).$$

Let

$$R = T_1 \cdot T_2 \cdot \ldots \cdot T_{\lceil \frac{n}{2} \rceil} = 0^m \cdot T_2 \cdot T_3 \cdot \ldots \cdot T_{\lceil \frac{n}{2} \rceil}.$$

Let V be the longest short zero-sum subsequence of TR^{-1} (if TR^{-1} has no short zero-sum subsequence then let V be the empty sequence).

Write $|V| = mq + t$ with $t \in [0, m-1]$. Clearly, $q \in [0, n-1]$. If $|VR| \geq mn$, then $0^{m-t} T_2 \cdots T_{n-q} V$ is a zero-sum subsequence of VR of length $|0^{m-t} T_2 \cdots T_{n-q} V| = mn$, a contradiction with $0 \notin \sum_{mn}(T)$. Therefore,

$$|VR| \leq mn - 1.$$

Hence,

$$|V| < mn/2.$$

If $T(RV)^{-1}$ has a short zero-sum subsequence V_1, then similarly to above we get $|V_1| < mn/2$, and so VV_1 is also a short zero-sum subsequence, but $|VV_1| > |V|$, a contradiction. Therefore, $T(RV)^{-1}$ contains no short zero-sum subsequence. Hence, $\eta(G) - 1 \geq |T(RV)^{-1}| = \mathsf{s}(G) - 1 - |RV| \geq \mathsf{s}(G) - mn$. □

Proof of Theorem 1.3. By Lemma 2.2, $\mathsf{s}(G) \geq \eta(G) + \exp(G) - 1$. So, it suffices to prove $\mathsf{s}(G) \leq \eta(G) + \exp(G) - 1$.

Let $m = \exp(G)$, and let $g_0 \in G$ such that $v_{g_0}(S) = \max_{g \in G}\{v_g(S)\}$. Let $h = v_{g_0}(S)$ and let $S_0 = -g_0 + S$. Then, $0 \notin \sum_m(S_0)$ follows from $0 \notin \sum_m(S)$, and $h = v_0(S_0) \geq \lfloor \frac{m-1}{2} \rfloor$. Write $S_0 = 0^h T$. Let T_0 be the maximal (in length) zero-sum subsequence of T of length $|T_0| \in [0, m]$. We assert that

$$|T_0| + h \leq m - 1 \text{ and } TT_0^{-1} \text{ contains no short zero-sum subsequence over } G.$$

If $|T_0| + h \geq m$ then $T_0 0^{m-|T_0|}$ is a zero-sum subsequence of S_0 of length m, a contradiction. If TT_0^{-1} contains a short zero-sum subsequence T_1, then $1 \leq |T_1| \leq |T_0|$ by the maximality of T_0. Therefore, $|T_0 T_1| \leq 2|T_0| \leq 2(m - 1 - h) \leq m$. Hence, $T_0 T_1$ is also a short zero-sum sequence over G. But, $|T_0 T_1| > |T_0|$, a contradiction on the maximality of $|T_0|$. This proves the assertion. Therefore, $\eta(G) - 1 \geq |TT_0^{-1}| = |S| - h - |T_0| = \mathsf{s}(G) - 1 - (h + |T_0|) \geq \mathsf{s}(G) - 1 - (m-1)$ and $\mathsf{s}(G) \leq \eta(G) + m - 1$ follows. □

Proof of Corollary 1.4. By Lemmas 2.7 and 2.5 we obtain that, $\eta(C_{mn} \oplus H) \leq (n+1)m + \mathsf{D}(H) - 2$. It follows from Lemma 2.4 that $\eta(C_{mn} \oplus H) \geq nm + 2(\mathsf{D}(H) - 1) = (n+1)m + \mathsf{D}(H) - 2$. Hence, $\eta(C_{mn} \oplus H) = (n+1)m + \mathsf{D}(H) - 2$.

Let $m_0 = \exp(H)$. Since $\exp(H)|m$, $m = m_0 m_1$ with $m_1 \in \mathbb{N}$. Let $n_0 = nm_1$. Then, $m_0 n_0 = mn$ and $n_0 \geq n \geq \max\{m|H| + 1, 4|H| + 2m\} \geq \max\{m_0|H| + 1, 4|H| + 2m_0\}$. It follows from Theorem 1.2 that $\mathsf{s}(C_{mn} \oplus H) = \mathsf{s}(C_{m_0 n_0} \oplus H) = m_0 n_0 + \eta(C_{m_0 n_0} \oplus H) - 1 = mn + \eta(C_{mn} \oplus H) - 1 = (2n+1)m + \mathsf{D}(H) - 3$. □

4. Concluding remarks

Let p be a prime, and let H be a finite abelian p-group. If $m = p^a = \mathsf{D}(H)$ for some positive integer a and if $n \geq \max\{m|H| + 1, 4|H| + 2m\}$ then m, n and H fulfill the condition of Corollary 1.4. Therefore,

$\mathsf{s}(C_{mn} \oplus H) = mn + \eta(C_{mn} \oplus H) - 1 = (2n+1)m + \mathsf{D}(H) - 3$. This result was first obtained by Geroldinger, Grynkiewicz and Schmid [20] for all $n \in \mathbb{N}$ by using a result in [35, Theorem 1.2], which was achieved with some highly technical results (see [35, Theorem 3.1]) on enumeration. It would be very interesting to find some pairs of $\{m, H\}$ with m not being a prime power to fulfill the condition of Corollary 1.4.

The equality $\mathsf{s}(G) = \eta(G) + \exp(G) - 1$ holds for all groups G for which $\mathsf{s}(G)$ is known. Here we give a list of these groups.

Proposition 4.1. *Let n, r, a, b, c, d, e, m be nonnegative integers with $n > 0$ and $r > 0$. Then the equality $\mathsf{s}(G) = \eta(G) + \exp(G) - 1$ holds true for the following groups G.*

1. $G = C_{n_1} \oplus C_{n_2}$ *with* $1 \le n_1 \mid n_2$.
2. $G = C_{2^a} \oplus C_{2^b}^{r-1}$, *where* $r \ge 2$, $b \ge 1$ *and* $a \in [1, b]$.
3. $G = C_{3^a 5^b}^3$, *where* $a + b \ge 1$.
4. $G = C_{3^a}^4$ *where* $a \ge 1$.
5. $G = C_{3\times 2^a}^3$ *where* $a \ge 1$.
6. *G is a p-group for some odd prime p with* $\mathsf{D}(G) = 2\exp(G) - 1$.
7. $G = H \oplus C_n$, *where p is an odd prime, H is a finite abelian p-group with* $\exp(H) \mid n = \exp(G)$ *and* $\mathsf{D}(H) \mid \exp(H)$.
8. $G = C_{mn}^3$ *with* $m = 3^a 5^b$ *and* $n = 7^c 11^d 13^e$ *such that*
$$m \ge \frac{2 \times 5^6 n^{17}}{(n^2-7)n - 64}.$$

Lemma 4.2 ([7, *Theorem 1.2*])**.** *Let n, r be positive integers. Then,*

1. $\eta(C_n^3) \ge 8n - 7$ *holds for all odd n.*
2. $\eta(C_n^4) \ge 19n - 18$ *holds for all odd n.*
3. $\eta(C_n^r) \ge (2^r - 1)(n - 1) + 1$.

Proof of Proposition 4.1. 1. It has been proved in [7, Theorem 1.1] that $\mathsf{s}(C_{n_1} \oplus C_{n_2}) = \eta(C_{n_1} \oplus C_{n_2}) + n_2 - 1 = 2n_1 + 2n_2 - 3$.

2. It has been proved in [7, Corollary 4.4] that $\mathsf{s}(G) = \eta(G) + 2^b - 1 = 2^{r-1}(2^a + 2^b - 2) + 1$.

3. Let $\ell = 3^a 5^b$. It has been proved in [16] that $\mathsf{s}(C_\ell^3) = 9\ell - 8$. Now the result follows from Lemmas 4.2 and 2.2.

4. Let $\ell = 3^a$. It has been proved in [10, Lemma 2.4] that $\mathsf{s}(C_\ell^4) = 20\ell - 19$. Now the result follows from Lemmas 4.2 and 2.2.

5. Let $\ell = 3 \times 2^a$. It has been proved in [16] that $\mathsf{s}(C_\ell^3) = 8\ell - 7$. Now the result follows from Lemmas 4.2 and 2.2.

6. It has been proved in [35, Theorem 1.2] that $\mathsf{s}(G) = \eta(G) + \exp(G) - 1 = 4\exp(G) - 3$.

7. The result follows from [20, Theorems 4.2.1 and 2].

8. Let $\ell = mn$. It has been proved in [10] that $\mathsf{s}(C_\ell^3) = 9\ell - 8$. Now the result follows from Lemmas 4.2 and 2.2. □

Since Conjecture 1.1 holds true for the groups G with $\exp(G) \in [3, 4]$ as mentioned in the Introduction, we do not include these groups in the list in Proposition 4.1. Conjecture 1.1 remains widely open. We even are unable to prove or disprove it for the group $G = C_5^r$ with $r \ge 4$.

Let $m = \exp(G)$, $\mathsf{s}_{km}(G)$ was first introduced by the second author in [14], and was studied further in [18,29]. For the case that $km = |G|$, $\mathsf{s}_{|G|}(G)$ has attracted a lot of attention. In 1996, the second author [13] proved the following result which can be regarded as an extension of the Erdős–Ginzburg–Ziv theorem.

$$\mathsf{s}_{|G|}(G) = |G| + \mathsf{D}(G) - 1. \tag{5}$$

Weighted generalizations of (5) have been made by many authors since 1996 (for example see [23,24,22,1,36]). A new proof for (5) has been provided recently in [5]. In fact, by using the method in [13] one can obtain that

$$\mathsf{s}_{km}(G) = km + \mathsf{D}(G) - 1 \tag{6}$$

provided that $km \geq |G|$.

Let $\ell(G)$ be the smallest integer t such that (6) holds true for all $k \geq t$. So, we have $\ell(G) \leq \frac{|G|}{m}$. For some further results and open problems on $\ell(G)$ we refer to [18].

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References

[1] S.D. Adhikari, Y.G. Chen, Davenport constant with weights and some related questions II, J. Combin. Theory Ser. A 115 (2008) 178–184.
[2] N. Alon, M. Dubiner, A lattice point problem and additive number theory, Combinatorica 15 (1995) 301–309.
[3] A. Bialostocki, Some problems in view of recent developments of the Erdős–Ginzburg–Ziv theorem, Integers 7 (2007) 10. #A07 (electronic).
[4] J. Bierbrauer, Y. Edel, Large caps in projective Galois spaces, Preprint.
[5] M. DeVos, L. Goddyn, B. Mohar, A generalization of Kneser's addition theorem, Adv. Math. 220 (2009) 1531–1548.
[6] Y. Edel, Sequences in abelian groups G of odd order without zero-sum subsequences of length $\exp(G)$, Des. Codes Cryptogr. 47 (2008) 125–134.
[7] Y. Edel, C. Elsholtz, A. Geroldinger, S. Kubertin, L. Rackham, Zero-sum problems in finite abelian groups and affine caps, Q. J. Math. 58 (2007) 159–186.
[8] Y. Edel, S. Ferret, I. Landjev, L. Storme, The classification of the largest caps in $AG(5, 3)$, J. Combin. Theory Ser. A 99 (2002) 95–110.
[9] P. Erdős, A. Ginzburg, A. Ziv, Theorem in the additive number theory, Bull. Res. Counc. Israel 10 (1961) 41–43.
[10] Y.S. Fan, W.D. Gao, Q.H. Zhong, On the Erdős–Ginzburg–Ziv constant of finite abelian groups of high rank, J. Number Theory 131 (2011) 1864–1874.
[11] P. Frankl, R.L. Graham, V. Rődl, On subsets of abelian groups with no three term arithmetic progression, J. Combin. Theory Ser. A 45 (1987) 157–161.
[12] W.D. Gao, Some problems in additive group theory and additive number theory, Ph.D. Thesis, Sichuan University, 1994.
[13] W.D. Gao, A combinatorial problem on finite abelian groups, J. Number Theory 58 (1996) 100–103.
[14] W.D. Gao, On zero-sum subsequences of restricted size, II, Discrete Math. 271 (2003) 51–59.
[15] W.D. Gao, A. Geroldinger, Zero-sum problems in finite abelian groups: a survey, Expo. Math. 24 (2006) 337–369.
[16] W.D. Gao, Q.H. Hou, W.A. Schmid, R. Thangadurai, On short zero-sum subsequences II, Integers 7 (2007) 22. Paper A21.
[17] W.D. Gao, R. Thangadurai, A variant of Kemnitz conjecture, J. Combin. Theory Ser. A 107 (2004) 69–86.
[18] W.D. Gao, R. Thangadurai, On zero-sum sequences of prescribed length, Aequationes Math. 72 (2006) 201–212.
[19] W.D. Gao, Y.X. Yang, Note on a combinatorial constant, J. Math. Res. Exposition 17 (1997) 139–140 (in Chinese).
[20] A. Geroldinger, D.J. Grynkiewicz, W.A. Schmid, Zero-sum problems with congruence conditions, Acta Math. Hungar. 131 (2011) 323–345.
[21] A. Geroldinger, F. Halter-Koch, Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory, in: Pure Appl. Math., vol. 278, Chapman & Hall/CRC, 2006.
[22] D. Grynkiewicz, A weighted Erdős–Ginzburg–Ziv theorem, Combinatorica 26 (2006) 445–453.
[23] Y.O. Hamidoune, On weighted sums in abelian groups, Discrete Math. 162 (1996) 127–132.
[24] Y.O. Hamidoune, A weighted generalization of Gao's $n + D - 1$ theorem, Combin. Probab. Comput. 17 (2008) 793–798.
[25] H. Harborth, Ein Extremalproblem für Gitterpunkte, J. Reine Angew. Math. 262 (1973) 356–360.
[26] R. Hill, On the largest size of cap in $S_{5,3}$, Atti Accad. Naz. Lincei Rend. 54 (1973) 378–384.
[27] R. Hill, Caps and groups, in: Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973) Tomo II, in: Atti dei Convegni Lincei, vol. 17, Accad. Naz. Lincei, Rome, 1976, pp. 389–394.
[28] A. Kemnitz, On a lattice point problem, Ars Combin. 16 (1983) 151–160.
[29] S. Kubertin, Zero-sums of length $kqkq$ in $\mathbb{Z}_q^d$, Acta Arith. 116 (2005) 145–152.
[30] R. Meshulam, On subsets of finite abelian groups with no 3-term arithmetic progressions, J. Combin. Theory Ser. A 71 (1995) 168–172.
[31] A. Potechin, Maximal caps in $AG(6, 3)$, Des. Codes Cryptogr. 46 (2008) 243–259.
[32] C. Reiher, On Kemnitz' conjecture concerning lattice points in the plane, Ramanujan J. 13 (2007) 333–337.
[33] L. Rònyai, On a conjecture of Kemnitz, Combinatorica 20 (2000) 569–573.
[34] S. Savchev, F. Chen, Long n-zero-free sequences in finite cyclic groups, Discrete Math. 308 (2008) 1–8.
[35] W.A. Schmid, J.J. Zhuang, On short zero-sum subsequences over p-groups, Ars Combin. 95 (2010) 343–352.
[36] X.N. Zeng, P.Z. Yuan, Weighted Davenport's constant and the weighted EGZ Theorem, Discrete Math. 311 (2011) 1940–1947.