REMARKS ON TINY ZERO-SUM SEQUENCES

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Abstract

Let G be an additive finite abelian group with exponent $\exp(G)$. Let $S = g_1 \cdot \ldots \cdot g_l$ be a sequence over G and $\mathsf{k}(S) = \mathrm{ord}(g_1)^{-1} + \cdots + \mathrm{ord}(g_l)^{-1}$ be its cross number. Let $\eta(G)$ (resp. $\mathsf{t}(G)$) be the smallest integer t such that every sequence of t elements (repetition allowed) from G contains a non-empty zero-sum subsequence T of length $|T| \le \exp(G)$ (resp. $\mathsf{k}(T) \le 1$). It is easy to see that $\mathsf{t}(G) \ge \eta(G)$ for all finite abelian groups G, and a previous result showed that for every positive integer $r \ge 4$, there exist finite abelian groups of rank r such that $\mathsf{t}(G) > \eta(G)$. In this paper we provide the first example of groups G of rank three with $\mathsf{t}(G) > \eta(G)$. We also prove that $\mathsf{t}(G) = \eta(G)$ for $G = C_2 \oplus C_{2p}$ where p is a prime.

1. Introduction

Let G be an additively written finite abelian group with $\exp(G)$ its exponent. A sequence $S = g_1 \cdot \ldots \cdot g_l$ over G is said to be a zero-sum sequence, if $\sum_{i=1}^{l} g_i = 0$. S is called a minimal zero-sum sequence, if it contains no proper zero-sum subsequence. The cross number $\mathsf{k}(S)$ of a sequence S is defined by

$$\mathsf{k}(S) = \sum_{i=1}^{l} \frac{1}{\mathrm{ord}(g_i)}.$$

The cross number is an important concept in factorization theory. For recent work on the cross number we refer to ([10], [12], [13]).

By $\mathsf{t}(G)$ we denote the smallest integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq t$ contains a non-empty zero-sum subsequence $S'|S$ with $\mathsf{k}(S') \leq 1$. Such a subsequence will be called a *tiny zero-sum subsequence*.

The study of $\mathsf{t}(G)$ goes back to the late 1980s, Lemke and Kleitman [17] proved that $\mathsf{t}(C_n) = n$, which confirmed a conjecture by Erdős and Lemke, where C_n denotes the cyclic group of n elements.

In the general case, Kleitman and Lemke [17] conjectured that $\mathsf{t}(G) \leq |G|$ holds for every finite abelian group G. This conjecture was confirmed by Geroldinger [9] in 1993. A different proof was found by Elledge and Hurlbert [4] in 2005 using graph pebbling. For more work on applications of graph pebbling to zero-sum problems we refer to ([2], [3], [15], [16]).

Quite recently, Girard [14] proved that, by using a result of Alon and Dubiner [1], for finite abelian groups of fixed rank, $\mathsf{t}(G)$ grows linearly in the exponent of G, which gives the correct order of magnitude.

Let $\eta(G)$ denote the smallest integer $t \in \mathbb{N}$ such that every sequence S over G of length $|S| \geq t$ contains a non-empty zero-sum subsequence $S'|S$ with $|S'| \leq \exp(G)$. Such a subsequence is called a *short zero-sum subsequence*. For more information on $\eta(G)$ we refer to [5] and [6].

Since $\mathsf{k}(T) \leq 1$ implies $|T| \leq \exp(G)$, we know that $\eta(G) \leq \mathsf{t}(G)$ always holds. The constant $\eta(G)$ is one of many classical invariants in so-called zero-sum theory. For zero-sum theory and its application, the interested reader is referred to [7] and [11].

Girard [14] noticed that if $\mathsf{t}(G) = \eta(G)$ for some finite abelian group G, then $\eta(H) \leq \eta(G)$ for any subgroup H of G, and then he deduced that for any positive integer $r \geq 4$, there is a finite abelian group of rank r such that $\mathsf{t}(G) > \eta(G)$. Girard [14] also proved that $\mathsf{t}(C_{p^\alpha}^2) = \eta(C_{p^\alpha}^2) = 3p^\alpha - 2$ for any prime p and conjectured that $\mathsf{t}(G) = \eta(G)$ for all finite abelian groups of rank two.

Conjecture 1.1. For all positive integers m, n with $m|n$, we have

$$\mathsf{t}(C_m \oplus C_n) = \eta(C_m \oplus C_n) = 2m + n - 2.$$

Conjecture 1.1 is wide open. For the case that G has rank three, he asked the following question.

Question 1.2. ([14], page 1856) Does $\mathsf{t}(G) = \eta(G)$ hold for all finite abelian groups G of rank three?

In this paper, we offer a negative answer to this question by showing

Theorem 1.3. *Let $n > 1$ be a positive integer, and let $G = C_2 \oplus C_2 \oplus C_{2n}$. Then $\mathsf{t}(G) > \eta(G) = 2n + 4$.*

We also prove the following.

Theorem 1.4. *Let p be a prime, and let $G = C_2 \oplus C_{2p}$. Then, $\mathsf{t}(G) = \eta(G)$.*

2. Notations and Preliminaries

Let $\mathbb{P}$ denote the set of prime numbers, $\mathbb{N}$ denote the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any two integers $a, b \in \mathbb{N}_0$, we set $[a, b] = \{x \in \mathbb{N}_0 : a \le x \le b\}$. Throughout this paper, all abelian groups will be written additively, and for $n, r \in \mathbb{N}$, we denote by C_n the cyclic group of order n, and denote by C_n^r the direct sum of r copies of C_n.

Let $\mathcal{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathcal{F}(G)$ are called sequences over G. We write sequences $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{v_g(S)}, \text{ with } v_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

We call $v_g(G)$ the multiplicity of g in S, and we say that S contains g if $v_g(S) > 0$. The unit element $1 \in \mathcal{F}(G)$ is called the empty sequence. A sequence S_1 is called a subsequence of S if $S_1 \mid S$ in $\mathcal{F}(G)$. For a subset A of G we denote $S_A = \Pi_{g \in A} g^{v_g(S)}$. If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$.

For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{v_g(S)} \in \mathcal{F}(G),$$

we call

- $|S| = l = \sum_{g \in G} v_g(G) \in \mathbb{N}_0$ the *length* of S,
- $\text{supp}(S) = \{g \in G | v_g(S) > 0\} \subset G$ the *support* of S,
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} v_g(S) g \in G$ the *sum* of S,

The sequence S is called *zero-sumfree* if it contains no nonempty zero-sum subsequence.

We denote by $\mathcal{A}(G) \subset \mathcal{F}(G)$ the set of all minimal zero-sum sequences over G. Every map of abelian groups $\varphi : G \to H$ extends to a homomorphism $\varphi : \mathcal{F}(G) \to$

$\mathcal{F}(H)$ where $\varphi(S) = \varphi(g_1) \cdot \ldots \cdot \varphi(g_l)$. If φ is a homomorphism then $\varphi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \ker(\varphi)$.

We shall use the following invariants on zero-sum sequences.

Definition 2.1. Let $n, t \in \mathbb{N}$ and $\exp(G) = n$. We denote by

- $\mathsf{D}(G)$ the smallest integer $t \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \geq t$ contains a non-empty zero-sum subsequence. The invariant $\mathsf{D}(G)$ is called the *Davenport constant* of G.

- $\mathsf{s}(G)$ the smallest integer $t \in \mathbb{N}$ such that every sequence $S \in \mathcal{F}(G)$ of length $|S| \geq t$ contains a zero-sum subsequence of length $\exp(G)$.

Throughout this paper, let p always denote an odd prime.

Lemma 2.2. *[11, Theorem 5.4.5] Let $n > 1$ be a positive integer, and let $S \in \mathcal{F}(C_n)$ be a sequence of length $n-1$. If S is zero-sumfree then $S = g^{n-1}$ for some generating element $g \in C_n$.*

Lemma 2.3. *Let $n > 1$ be a positive integer, and let $S \in \mathcal{F}(C_n)$ be a sequence of length $2n-1$. If S contains no two disjoint nonempty zero-sum subsequences then $S = g^{2n-1}$ for some generating element $g \in C_n$.*

Proof. Let T be an arbitrary subsequence of S of length $|T| = n-1$. Then, $|ST^{-1}| = n = \mathsf{D}(C_n)$. Therefore, ST^{-1} contains a zero-sum subsequence. It follows from the hypothesis of the lemma that T is zero-sumfree. Hence, $T = g^{n-1}$ for some generating element $g \in C_n$ by Lemma 2.2. Now the result follows from the arbitrariness of the choice of T. □

Lemma 2.4. *Let $n > 1$ be a positive integer, $G = C_2 \oplus C_2 \oplus C_{2n}$, and let (e_1, e_2, e_3) be a basis of G with $\mathrm{ord}(e_1) = \mathrm{ord}(e_2) = 2$ and $\mathrm{ord}(e_3) = 2n$. Then, the sequence $S = e_3^{2n-1} e_1 e_2 (e_1+e_3)(e_1+e_2+e_3)(e_1+e_2)$ contains no tiny zero-sum subsequence and therefore $\mathsf{t}(G) > 2n+4$.*

Proof. Let $W = e_3^{2n-1} e_1 e_2 (e_1+e_3)(e_1+e_2+e_3)$. It is easy to see that W contains no short zero-sum subsequence, and the sequence $W_1 = e_1 e_2 (e_1+e_2)$ is the only short zero-sum subsequence of S. But $\mathsf{k}(W_1) = \frac{3}{2} > 1$. Therefore, S contains no tiny zero-sum subsequence. Hence, $\mathsf{t}(G) > |S| = 2n+4$. □

3. Proof of the Main Results

Proof of Theorem 1.3. By Lemma 2.4, it suffices to prove that $\eta(C_2 \oplus C_2 \oplus C_{2n}) = 2n+4$. Let (e_1, e_2, e_3) be a basis of G with $\mathrm{ord}(e_1) = \mathrm{ord}(e_2) = 2$ and $\mathrm{ord}(e_3) = 2n$. Let $W = e_3^{2n-1} e_1 e_2 (e_1+e_3)(e_1+e_2+e_3)$. Clearly, W contains no short zero-sum subsequence. Therefore, $\eta(G) \geq 1 + |W| = 2n+4$.

So, it remains to prove $\eta(G) \le 2n+4$. Let $S \in \mathcal{F}(G)$ be a sequence of length $2n+4$, we need to show that S contains a short zero-sum subsequence. Assume to the contrary that S contains no short zero-sum subsequence.

Let $\varphi : G \to C_2^3$ be the homomorphism with $\ker(\varphi) = C_n$. Since $\eta(C_2^3) = 8$ and $|S| = 2n+4$, S has a decomposition $S = T_1 \cdots T_{n-1}T$ with $\sigma(T_i) \in \ker(\varphi) \setminus \{0\}$ and $|T_i| \le 2$ for each $i \in [1, n-1]$. It follows that $|T| \ge |S| - 2(n-1) \ge 6$.

Since S contains no short zero-sum subsequence and $\mathsf{D}(\ker(\varphi)) = \mathsf{D}(C_n) = n$, the sequence $\sigma(T_1)\cdot\ldots\cdot\sigma(T_{n-1})$ is zero-sumfree over C_n and $\varphi(T)$ contains no short zero-sum subsequence over C_2^3. It follows that $|\operatorname{supp}(\varphi(T))| = |\varphi(T)| = |T|$. Recall that $|T| \ge 6$. Let T' be a subsequence of T of length $|T'| = 6$. So, $\varphi(T')$ is a subset of C_2^3. Suppose that $(C_2^3 \setminus \{0\}) \setminus \varphi(T') = \{\alpha\}$. Let x_1, x_2, x_3 be a basis of C_2^3. Then, $C_2^3 = \{0, x_1, x_2, x_3, x_1+x_2, x_1+x_3, x_2+x_3, x_1+x_2+x_3\}$. Let ψ be an automorphism over C_2^3 with $\psi(\alpha) = x_1 + x_2 + x_3$. Then, $\varphi(T') = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ with $\alpha_1 = \psi^{-1}(x_1), \alpha_2 = \psi^{-1}(x_2), \alpha_3 = \psi^{-1}(x_3), \alpha_4 = \psi^{-1}(x_1 + x_2), \alpha_5 = \psi^{-1}(x_1 + x_3)$ and $\alpha_6 = \psi^{-1}(x_2+x_3)$. It is easy to check that the following subsequences of $\varphi(T')$ are all zero-sum.

$$\alpha_1\alpha_2\alpha_4, \alpha_1\alpha_3\alpha_5, \alpha_2\alpha_3\alpha_6, \alpha_4\alpha_5\alpha_6, \alpha_1\alpha_2\alpha_5\alpha_6, \alpha_1\alpha_3\alpha_4\alpha_6, \alpha_2\alpha_3\alpha_4\alpha_5. \quad (1)$$

Let $T' = g_1g_2g_3g_4g_5g_6$ with $\varphi(g_i) = \alpha_i$ for every $i \in [1, 6]$. From (1) we know that each of the following subsequences of T' has sum in $\ker(\varphi)$ and each is of length in $[3, 4]$:

$$g_1g_2g_4, g_1g_3g_5, g_2g_3g_6, g_4g_5g_6, g_1g_2g_5g_6, g_1g_3g_4g_6, g_2g_3g_4g_5. \quad (2)$$

Let T_n be any sequence listed in (2). Then, $\sigma(T_1)\cdot\ldots\cdot\sigma(T_{n-1})\cdot\sigma(T_n)$ is a sequence over $\ker(\varphi) = C_n$ of length n. If there is a subset $I \subset [1, n]$ such that $1 \le |I| \le n-1$ and such that $\sum_{i\in I} \sigma(T_i) = 0$, then $\prod_{i\in I} T_i$ is a zero-sum subsequence of S of length

$$\Big|\prod_{i\in I} T_i\Big| \le 2(|I| - 1) + 4 \le 2(n-2) + 4 = 2n,$$

a contradiction. Therefore, every subsequence of $\sigma(T_1)\cdot\ldots\cdot\sigma(T_{n-1})\cdot\sigma(T_n)$ of length $n-1$ is zero-sumfree. Therefore, $\sigma(T_1) = \sigma(T_2) = \cdots = \sigma(T_n)$ by Lemma 2.2. This proves that every sequence listed in (2) has sum $\sigma(T_1)$. Therefore,

$$\begin{aligned} g_1 + g_2 + g_4 &= g_1 + g_3 + g_5 = g_2 + g_3 + g_6 = g_4 + g_5 + g_6 \\ &= g_1 + g_2 + g_5 + g_6 = g_1 + g_3 + g_4 + g_6 = g_2 + g_3 + g_4 + g_5. \end{aligned}$$

From $g_1+g_2+g_4 = g_1+g_2+g_5+g_6$ we get that $g_4 = g_5+g_6$. Similarly, we obtain that $g_5 = g_4+g_6$ and $g_6 = g_4+g_5$. Therefore, $g_4+g_5+g_6 = (g_5+g_6)+(g_4+g_6)+(g_4+g_5)$ and $g_4 + g_5 + g_6 = 0$ follows. Hence, $g_4g_5g_6$ is a short zero-sum subsequence of S, a contradiction. This proves that $\eta(C_2 \oplus C_2 \oplus C_{2n}) = 2n + 4$. □

Proof of Theorem 1.4. As mentioned in the introduction we always have $\mathsf{t}(G) \ge \eta(G)$. From a result in [11, Theorem 5.8.3] we know that $\mathsf{t}(G) \ge \eta(G) = 2p+2$. So, it remains to prove that $\mathsf{t}(G) \le 2p+2$. Let $S \in \mathcal{F}(G)$ be of length $|S| = 2p+2$. We need to show that S contains a tiny zero-sum subsequence. Assume to the contrary that S contains no tiny zero-sum subsequence.

For every integer d, let S_d denote the subsequence of S consisting of all terms of S of order d. Then $S = S_2 S_p S_{2p}$ and

$$|S_2| + |S_p| + |S_{2p}| = 2p + 2. \tag{3}$$

Let $\varphi : G \to C_2^2$ be the homomorphism with $\ker(\varphi) = C_p$ and $\psi : G \to C_p$ be the homomorphism with $\ker(\psi) = C_2^2$. For any element $g \mid S_2$ we have $g \in \ker(\psi)$ and since $\eta(C_2^2) = 4$ we deduce that

$$|S_2| \le 3. \tag{4}$$

Similarly, as $\eta(C_p) = p$ we obtain that $|S_p| \le p - 1$ and $|S_{2p}| \ge p$ follows.

Since $\eta(\varphi(G)) = \eta(C_2^2) = 4$, S_{2p} has a decomposition

$$S_{2p} = T_1 \cdots T_m T$$

with $|T_i| = 2, \sigma(T_i) \in \ker(\varphi) = C_p$ for every $i \in [1, m]$ and $|T| \le 3$.

If there is a short zero-sum subsequence of $\sigma(T_1) \cdot \ldots \cdot \sigma(T_m) \cdot S_p$, i.e., there is a subset $I \subset [1, m]$ and a subsequence $T_0 | S_p$ such that $T_0 \prod_{i \in I} \sigma(T_i)$ is a short zero-sum sequence over C_p, then $T_0 \prod_{i \in I} T_i$ is a zero-sum subsequence of S with $\mathsf{k}(T_0 \prod_{i \in I} T_i) = \mathsf{k}(T_0) + \sum_{i \in I} \mathsf{k}(T_i) = \frac{|T_0|}{p} + \frac{|I|}{p} \le 1$, a contradiction. Therefore, $\sigma(T_1) \cdot \ldots \cdot \sigma(T_m) \cdot S_p$ contains no short zero-sum subsequence over $\ker(\varphi) = C_p$. It follows that

$$m + |S_p| \le \eta(C_p) - 1 = p - 1. \tag{5}$$

From $|T| \le 3$ and $|T_i| = 2$ we derive that $m \ge \frac{|S_{2p}| - 3}{2}$. This together with (5) gives that

$$|S_{2p}| + 2|S_p| \le 2p + 1. \tag{6}$$

By (3), (4), and (6) we obtain that

$$2p + 2 - 3 + |S_p| \le |S| - |S_2| + |S_p| = |S_{2p}| + 2|S_p| \le 2p + 1$$

and so $|S_p| \le 2$. Hence, $|S_{2p}| \ge 2p - 3 \ge \eta(C_p)$. Therefore, there exists a subsequence $R \mid S_{2p}$ such that $\sigma(R) \in \ker(\psi)$ and $|R| \le p$. It follows that $\mathsf{k}(R) = \frac{|R|}{2p} \le \frac{1}{2}$. If $|S_2| = 3$ then the sequence $\sigma(R) \cdot S_2 \in \mathcal{F}(C_2^2)$ is of length 4, and it follows from $\eta(C_2^2) = 4$ that the sequence $\sigma(R) \cdot S_2$ contains a short zero-sum subsequence W over C_2^2. By the contradiction hypothesis we must have that W is of the form $\sigma(R)g$ where g is a term of S. So, $R \cdot g$ is a zero-sum subsequence of S with $\mathsf{k}(W) = \mathsf{k}(R) + \mathsf{k}(g) \le 1$, a contradiction. Therefore, $|S_2| \le 2$. Similarly to above, by (3) and (6) we deduce that $|S_p| \le 1$ and $|S_{2p}| \ge 2p - 1$.

We show next that $|S_2| \le 1$. Assume to the contrary that $|S_2| = 2$. We assert that $\psi(S_{2p})$ contains no two disjoint short zero-sum subsequences over $\psi(G) =$

C_p. Otherwise, there exist two disjoint subsequences W_1, W_2 of S_{2p} such that $\sigma(W_1), \sigma(W_2) \in \ker(\psi) = C_2^2$ and $\mathsf{k}(W_1) \le \frac{1}{2}$, $\mathsf{k}(W_2) \le \frac{1}{2}$. Now the sequence $\sigma(W_1) \cdot \sigma(W_2) \cdot S_2 \in \mathcal{F}(C_2^2)$ is of length 4. Similarly to the case that $|S_2| = 3$ we can find a tiny zero-sum subsequence of $W_1W_2S_2$, a contradiction. It follows from $\eta(\psi(G)) = \eta(C_p) = p$ and $|S_{2p}| \ge 2p-1$ that every subsequence of $\psi(S_{2p})$ of length $p-1$ is zero-sumfree. Therefore, $\psi(S_{2p}) = \beta^{|S_{2p}|}$ for some $\beta \in \psi(G) = C_p$ by Lemma 2.3. Let W' be any subsequence of S_{2p} of length p. Then, $\sigma(W') \in \ker(\psi) = C_2^2$. Let $C_2^2 \setminus \{0, \operatorname{supp}(S_2)\} = \{y\}$. Since S contains no tiny zero-sum subsequence, similarly to above we infer that $\sigma(W') = y$. By the arbitrariness of the choice of W' we obtain that $S_{2p} = g^{|S_{2p}|}$. Now m in equation (5) can be chosen satisfying $m \ge \frac{|S_{2p}|-1}{2}$ and therefore the equation (6) can be improved to $|S_{2p}|+2|S_p| \le 2p-1$. But the above inequality is impossible as $|S_{2p}| + |S_p| = 2p + 2 - |S_2| = 2p$. This proves that $|S_2| \le 1$. It follows from equation (6) and (3) that

$$|S_2| = 1, |S_p| = 0 \text{ and } |S_{2p}| = 2p + 1.$$

By (5) we have that $p-1 = \frac{2p+1-3}{2} = \frac{|\varphi(S_{2p})|-3}{2} \le m \le p-1$. Therefore, $m = p-1$. Since S contains no tiny zero-sum subsequence, the sequence $\sigma(T_1) \cdot \ldots \cdot \sigma(T_{p-1})$ is a zero-sumfree sequence over $\ker(\varphi) = C_p$. It follows from Lemma 2.2 that

$$\sigma(T_1) = \cdots = \sigma(T_{p-1}) = h$$

for some $h \in \ker(\varphi) = C_p$.

Let $S(T_1 \cdots T_{p-1})^{-1} = g_0g_1g_2g_3$ with $S_2 = g_0$. Since S contains no tiny zero-sum subsequence, it follows from $m = p-1$ and $\eta(C_p) = p$ that $\varphi(g_1g_2g_3)$ contains no short zero-sum subsequence over C_2^2. Therefore, $\varphi(g_1)$, $\varphi(g_2)$ and $\varphi(g_3)$ are distinct in $C_2^2 \setminus \{0\}$. Without loss of generality we assume that $\varphi(g_0) = \varphi(g_1) = \varphi(g_2) + \varphi(g_3)$. Let $U_1 = g_0g_1$, $U_2 = g_0g_2g_3$, $U_3 = g_1g_2g_3$. Then $\sigma(U_i) \in \ker(\varphi)$ for every $i \in [1,3]$. So, for every $i \in [1,3]$, the sequence $\sigma(T_1) \cdot \ldots \cdot \sigma(T_{p-1}) \cdot \sigma(U_i) = h^{p-1}\sigma(U_i)$ contains a zero-sum subsequence V_i over $\ker(\varphi)$, i.e., there exists a subset $J_i \subseteq [1, p-1]$ such that $V_i = (\Pi_{j \in J_i} \sigma(T_j)) \cdot \sigma(U_i)$ for each $i \in [1,3]$. Let $X_i = \Pi_{j \in J_i} T_j \cdot U_i$ for each $i \in [1,3]$. Then, X_1, X_2 and X_3 are zero-sum subsequences of S. Let $t_i = |J_i|$ for each $i \in [1,3]$. Then,

$$V_1 = h^{t_1}(g_0 + g_1),\ V_2 = h^{t_2}(g_0 + g_2 + g_3),\ V_3 = h^{t_3}(g_1 + g_2 + g_3).$$

Since $\mathsf{k}(X_i) > 1$ for every $i \in [1,3]$, by a straightforward computation we obtain that $\frac{p+1}{2} \le t_1 \le p-1$, $\frac{p-1}{2} \le t_2 \le p-1$ and $t_3 = p-1$. Therefore,

$$p + 1 \le t_1 + t_2 + 1 \le 2p - 1. \tag{7}$$

From V_i is zero-sum over $\ker(\varphi) = C_p$ we infer that

$$t_1h + g_0 + g_1 = t_2h + g_0 + g_2 + g_3 = (p-1)h + g_1 + g_2 + g_3 = 0.$$

Therefore,

$$(t_1h + g_0 + g_1) + (t_2h + g_0 + g_2 + g_3) - ((p-1)h + g_1 + g_2 + g_3) = 0.$$

This together with $2g_0 = 0$ gives that $(t_1 + t_2 + 1)h = 0 \in \ker(\varphi) = C_p$. Therefore, $t_1 + t_2 + 1 \equiv 0 \pmod p$, a contradiction to (7). This completes the proof. □

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