LONG SETS OF LENGTHS WITH MAXIMAL ELASTICITY

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Abstract. We introduce a new invariant describing the structure of sets of lengths in atomic monoids and domains. For an atomic monoid H, let $\Delta_\rho(H)$ be the set of all positive integers d which occur as differences of arbitrarily long arithmetical progressions contained in sets of lengths having maximal elasticity $\rho(H)$. We study $\Delta_\rho(H)$ for transfer Krull monoids of finite type (including commutative Krull domains with finite class group) with methods from additive combinatorics, and also for a class of weakly Krull domains (including orders in algebraic number fields) for which we use ideal theoretic methods.

1. Introduction

Let H be a monoid or domain such that every (non-zero and non-unit) element can be written as a finite product of atoms. If $a = u_1 \cdot \ldots \cdot u_k$ is a factorization into atoms $u_1, \ldots, u_k$, then k is called the length of this factorization and the set $\mathsf L(a) \subset \mathbb N$ of all possible factorization lengths is called the set of lengths of a. The system $\mathcal L(H) = \{\mathsf L(a) \mid a \in H\}$ of all sets of lengths is a well-studied means of describing the non-uniqueness of factorizations of H. If there is some $a \in H$ such that $|\mathsf L(a)| > 1$, then $\mathsf L(a^n) \supset \mathsf L(a) + \ldots + \mathsf L(a)$ whence $\mathsf L(a^n)$ has more than n elements for every $n \in \mathbb N$. Weak ideal theoretic conditions on H guarantee that all sets of lengths are finite. Then, apart from the trivial case where all sets of lengths are singletons, $\mathcal L(H)$ is a family of finite subsets of the integers containing arbitrarily long sets. Only in a couple of very special cases the system $\mathcal L(H)$ can be written down explicitly. In general, $\mathcal L(H)$ is described by parameters such as the set of distances $\Delta(H)$, the elasticity $\rho(H)$, and others. We recall the definition of the elasticity $\rho(H)$. If $L \in \mathcal L(H)$, then $\rho(L) = \sup(L)/\min L$ is the elasticity of L (thus $\rho(L) = 1$ if and only if $|L| = 1$). The elasticity $\rho(H)$ of H is the supremum of all $\rho(L)$ over all $L \in \mathcal L(H)$, and we say that it is accepted if there is some $L \in \mathcal L(H)$ such that $\rho(H) = \rho(L) < \infty$.

The goal of the present paper is to study the possible differences of arbitrarily long arithmetical progressions contained in sets of lengths having maximal possible elasticity. More precisely, suppose that H has accepted elasticity with $1 < \rho(H) < \infty$. Then let $\Delta_\rho(H)$ denote the set of all $d \in \mathbb N$ with the following property: for every $k \in \mathbb N$, there is some $L_k \in \mathcal L(H)$ with $\rho(L_k) = \rho(H)$ and $L_k = y_k + \big(L_k' \cup \{0, d, \ldots, \ell_k d\} \cup L_k''\big) \subset y_k + d\mathbb Z$, where $y_k \in \mathbb Z$, $\max L_k' < 0$, $\min L_k'' > \ell_k d$, and $\ell_k \ge k$. We study $\Delta_\rho(H)$ for transfer Krull monoids of finite type and for classes of weakly Krull monoids.

A transfer Krull monoid of finite type is a monoid having a weak transfer homomorphism to a monoid of zero-sum sequences over a finite subset of an abelian group. Transfer homomorphisms preserve factorization lengths which implies that the systems of sets of lengths of the two monoids coincide. This setting includes commutative Krull domains with finite class group, but also classes of not necessarily integrally closed noetherian domains, and classes of non-commutative Dedekind prime rings (for a detailed discussion see the beginning of Section 3).

Let H be a transfer Krull monoid over a finite abelian group G such that $|G| \ge 3$. Then $\mathcal L(H) = \mathcal L\big(\mathcal B(G)\big) =: \mathcal L(G)$, whence sets of lengths of H can be studied in the monoid $\mathcal B(G)$ of zero-sum sequences over G and methods from additive combinatorics can be applied. This setting has found wide interest in the literature ([8, 17, 34]). Our main results on $\Delta_\rho(\cdot)$ for transfer Krull monoids are summarized

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after Conjecture 3.20. In a discussion preceding Lemma 3.2 we review the tools from zero-sum theory required for studying $\Delta_\rho(\cdot)$ and their state of the art. A central question in all studies of systems of sets of lengths is the so-called Characterization Problem, which asks whether for two non-isomorphic finite abelian groups G and G' (with Davenport constant $\mathsf D(G) \ge 4$) the systems of sets of lengths $\mathcal L(G)$ and $\mathcal L(G')$ can coincide. The standing conjecture is that this is not possible (see [15, Section 6] for a survey, and [20, 25, 37] for recent progress), and the new invariant $\Delta_\rho(\cdot)$ turns out to be a further useful tool in these investigations (Corollary 3.19).

Within factorization theory the case of (transfer) Krull monoids and domains is by far the best understood case. Much less is known in the non-Krull case. The most investigated class are Mori domains R with non-zero conductor $\mathfrak f$, finite v-class group, and a finiteness condition on the factor ring $R/\mathfrak f$ (see [13, 31]). However, in the overwhelming number of situations only abstract arithmetical finiteness results are known but no precise results (such as in the Krull case). Mori domains, which are weakly Krull, have a defining family of one-dimensional local Mori domains which provides a strategy for obtaining precise results. In Section 4 we study $\Delta_\rho(\cdot)$ for such weakly Krull Mori domains and for their monoids of v-invertible v-ideals under natural algebraic finiteness assumptions which are satisfied, among others, by orders in algebraic number fields (Theorem 4.4). This is done by studying the local case first and then the local results are glued together with the help of the associated T-block monoid. Our results on $\Delta_\rho(\cdot)$ allow to reveal further classes of weakly Krull monoids which are not transfer Krull (Corollary 4.6).

2. Background on sets of lengths

For integers a and b, we denote by $[a, b] = \{x \in \mathbb Z \mid a \le x \le b\}$ the discrete interval between a and b. Let $L \subset \mathbb Z$ be a subset. If $d \in \mathbb N$ and $\ell, M \in \mathbb N_0$, then L is called an *almost arithmetical progression* (AAP for short) with difference d, length ℓ, and bound M if

$$L = y + (L' \cup \{0, d, \ldots, \ell d\} \cup L'') \subset y + d\mathbb Z \tag{2.1}$$

where $y \in \mathbb Z$, $L' \subset [-M, -1]$, and $L'' \subset \ell d + [1, M]$. If $L' \subset \mathbb Z$, then $L + L' = \{a + b \mid a \in L, b \in L'\}$ denotes the sumset. If $L = \{m_1, \ldots, m_k\} \subset \mathbb Z$ is finite with $k \in \mathbb N_0$ and $m_1 < \ldots < m_k$, then $\Delta(L) = \{m_i - m_{i-1} \mid i \in [2, k]\} \subset \mathbb N$ denotes the set of distances of L. If $L \subset \mathbb N$ is a subset of the positive integers, then $\rho(L) = \sup L / \min L$ denotes its elasticity, and for convenience we set $\rho(\{0\}) = 1$.

Let G be a finite abelian group. Let $r \in \mathbb N$ and $(e_1, \ldots, e_r)$ be an r-tuple of elements of G. Then $(e_1, \ldots, e_r)$ is said to be independent if $e_i \ne 0$ for all $i \in [1, r]$ and if for all $(m_1, \ldots, m_r) \in \mathbb Z^r$ an equation $m_1 e_1 + \ldots + m_r e_r = 0$ implies that $m_i e_i = 0$ for all $i \in [1, r]$. Furthermore, $(e_1, \ldots, e_r)$ is said to be a basis of G if it is independent and $G = \langle e_1 \rangle \oplus \ldots \oplus \langle e_r \rangle$. For every $n \in \mathbb N$, we denote by C_n an additive cyclic group of order n.

By a *monoid*, we mean an associative semigroup with unit element, and if not stated otherwise we use multiplicative notation. Let H be a monoid with unit-element $1 = 1_H \in H$. We denote by $H^\times$ the group of invertible elements and say that H is reduced if $H^\times = \{1\}$. Let $S \subset H$ be a subset and $a \in S$. Then $[S] \subset H$ denotes the submonoid generated by S, and $[a] = [\{a\}] = \{a^k \mid k \in \mathbb N_0\}$ is the submonoid generated by a. We say that the subset S is divisor-closed if $a, b \in H$ and $ab \in S$ implies that $a, b \in S$. We denote by $[\![S]\!]$ the smallest divisor-closed submonid containing S, and $[\![a]\!] = [\![\{a\}]\!]$ is the smallest divisor-closed submonoid of H containing a. The monoid H is said to be unit-cancellative if for each two elements $a, u \in H$ any of the equations $au = a$ or $ua = a$ implies that $u \in H^\times$. Clearly, every cancellative monoid is unit-cancellative.

Suppose that H is unit-cancellative. An element $u \in H$ is said to be irreducible (or an atom) if $u \notin H^\times$ and any equation of the form $u = ab$, with $a, b \in H$, implies that $a \in H^\times$ or $b \in H^\times$. Let $\mathcal A(H)$ denote the set of atoms, and we say that H is atomic if every non-unit is a finite product of atoms. If H satisfies the ascending chain condition on principal left ideals and on principal right ideals, then H is atomic ([11, Theorem 2.6]). If $a \in H \setminus H^\times$ and $a = u_1 \cdot \ldots \cdot u_k$, where $k \in \mathbb N$ and $u_1, \ldots, u_k \in \mathcal A(H)$, then k is a factorization length of a, and

$$\mathsf{L}_H(a) = \mathsf{L}(a) = \{k \mid k \text{ is a factorization length of } a\} \subset \mathbb{N}$$

denotes the *set of lengths* of a. It is convenient to set $\mathsf{L}(a) = \{0\}$ for all $a \in H^\times$. The family

$$\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$$

is called the *system of sets of lengths* of H, and

$$\rho(H) = \sup\{\rho(L) \mid L \in \mathcal{L}(H)\} \in \mathbb{R}_{\ge 1} \cup \{\infty\}$$

denotes the *elasticity* of H. We say that a monoid H has *accepted elasticity*

- if it is atomic unit-cancellative with elasticity $\rho(H) < \infty$, and there is an $L \in \mathcal{L}(H)$ such that $\rho(L) = \rho(H)$.

Let H be a monoid with accepted elasticity. Then $\sup L < \infty$ for every $L \in \mathcal{L}(H)$ and for a subset $S \subset H$,

$$\Delta_H(S) = \bigcup_{a \in S} \Delta(\mathsf{L}_H(a)) \ \subset \mathbb{N}$$

denotes the set of distances of S. Let $S \subset H$ be a divisor-closed submonoid and $a \in S$. Then $S^\times = H^\times$, $\mathcal{A}(S) = \mathcal{A}(H)$, $\mathsf{L}_S(a) = \mathsf{L}_H(a)$, and $\mathcal{L}(S) \subset \mathcal{L}(H)$. Furthermore, we have $\Delta_S(S) = \Delta_H(S)$ and we set $\Delta(S) = \Delta_S(S)$ and $\Delta(H) = \Delta_H(H)$. By definition we have $\Delta(H) = \emptyset$ if and only if $\rho(H) = 1$.

For any set P, we denote by $\mathcal{F}(P)$ the free abelian monoid with basis P. If

$$a = \prod_{p \in P} p^{\mathsf{v}_p(a)} \in \mathcal{F}(P), \quad \text{where} \quad \mathsf{v}_p \colon \mathcal{F}(P) \to \mathbb{N}_0 \quad \text{is the } p\text{-adic exponent}\,,$$

then $|a| = \sum_{p \in P} \mathsf{v}_p(a) \in \mathbb{N}_0$ is the length of a. Let D be a monoid. A submonoid $H \subset D$ is said to be *saturated* if $a \in D$, $b \in H$, and ($ab \in H$ or $ba \in H$) imply that $a \in H$. A commutative monoid H is Krull if its associated reduced monoid is a saturated submonoid of a free abelian monoid ([17, Theorem 2.4.8]). A commutative domain is Krull if and only if its monoid of non-zero elements is a Krull monoid. The theory of commutative Krull monoids and domains is presented in [28, 17].

Let G be an additive abelian group and $G_0 \subset G$ a subset. An element

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G_0} g^{\mathsf{v}_g(S)} \in \mathcal{F}(G_0)$$

is said to be a zero-sum sequence if its sum $\sigma(S) = g_1 + \ldots + g_\ell = \sum_{g \in G_0} \mathsf{v}_g(S) g$ equals zero. Then the set $\mathcal{B}(G_0)$ of all zero-sum sequences over G_0 is a submonoid, and since $\mathcal{B}(G_0) \subset \mathcal{F}(G_0)$ is saturated, it is a commutative Krull monoid. If S is as above, then $|S| = \ell \in \mathbb{N}_0$ is the length of S and $\operatorname{supp}(S) = \{g_1, \ldots, g_\ell\} \subset G$ denotes its support. The monoid $\mathcal{B}(G_0)$ plays a crucial role in Section 3. It is usual to set $\mathcal{L}(G_0) := \mathcal{L}\big(\mathcal{B}(G_0)\big)$, $\mathcal{A}(G_0) := \mathcal{A}\big(\mathcal{B}(G_0)\big)$, $\rho(G_0) := \rho\big(\mathcal{B}(G_0)\big)$, and $\Delta(G_0) := \Delta\big(\mathcal{B}(G_0)\big)$ (although this is an abuse of notation, it will never lead to confusion). If G_0 is finite, then $\mathcal{A}(G_0)$ is finite and

$$\mathsf{D}(G_0) = \max\{|U| \mid U \in \mathcal{A}(G_0)\} \in \mathbb{N}$$

denotes the *Davenport constant* of G_0.

Now we introduce the new arithmetical invariant, $\Delta_\rho(\cdot)$, to be studied in the present paper. For convenience we repeat the definition of the well-studied invariant $\Delta_1(\cdot)$ ([17, Definition 4.3.12]).

Definition 2.1. Let H be an atomic unit-cancellative monoid.

1. Let $\Delta_1(H)$ denote the set of all $d \in \mathbb{N}$ having the following property:
   For every $k \in \mathbb{N}$, there is some $L_k \in \mathcal{L}(H)$ which is an AAP with difference d and length at least k.
2. Let $\Delta_\rho(H)$ denote the set of all $d \in \mathbb{N}$ having the following property:
   For every $k \in \mathbb{N}$, there is some $L_k \in \mathcal{L}(H)$ which is an AAP with difference d, length at least k, and with $\rho(L_k) = \rho(H)$.

3. We set $\Delta_\rho^*(H) = \{\min \Delta_H([a]) \mid a \in H \text{ with } \rho(\mathsf L(a)) = \rho(H)\}$.

By definition, we have

$$\Delta_\rho(H) \subset \Delta_1(H) \subset \Delta(H)\,, \tag{2.2}$$

and $\Delta_\rho(H) = \emptyset$ if H does not have accepted elasticity.

The set $\Delta_1(H)$ is studied with the help of the set $\Delta^*(H)$ which is defined as the set of all $d \in \mathbb N$ having the following property ([17, Definition 4.3.12]):

There is a divisor-closed submonoid $S \subset H$ with $\Delta(S) \ne \emptyset$ and $d = \min \Delta(S)$.

If H is a commutative cancellative BF-monoid, then, by [17, Proposition 4.3.14],

$$\Delta^*(H) = \{\min \Delta(\llbracket a \rrbracket) \mid a \in H \text{ with } \Delta(\llbracket a \rrbracket) \ne \emptyset\}\,. \tag{2.3}$$

The sets $\Delta^*(H)$, called the set of minimal distances of H, and $\Delta_1(H)$ have found wide attention, so far mainly for transfer Krull monoids over finite abelian groups ([24, 20, 25, 37, 32]).

In the present paper we study $\Delta_\rho(H)$, and the set $\Delta_\rho^*(H)$ is a technical tool to do so. The relationship between the two sets is the topic of Lemma 2.4. In particular, we have $\emptyset \ne \Delta_\rho^*(H) \subset \Delta_\rho(H)$ (provided that H has accepted elasticity $\rho(H) > 1$). Equations (2.3) and (2.4) reveal the formal correspondence between $\Delta^*(H)$ and $\Delta_\rho^*(H)$ in the case of commutative monoids. However, there exist commutative monoids H and divisor-closed submonoids $S \subset H$ with $\rho(S) = \rho(H) > 1$ such that $\min \Delta(S) \notin \Delta_\rho(H)$ (use Theorem 3.5 with $S = H \in \{\mathcal B(C_4), \mathcal B(C_6), \mathcal B(C_{10})\}$). Thus, in contrast to (2.3), in Equation (2.4) we cannot replace $\llbracket a \rrbracket$ by an arbitrary divisor-closed submonoid.

In contrast to the formal similarity in the definitions, the invariants $\Delta_\rho(H)$ and $\Delta_1(H)$ show a very different behavior (in particular for transfer Krull monoids over finite abelian groups, see Section 3). Thus the additional requirement on the elasticity is a very strong one.

We start with a technical lemma analysing the set $\Delta_\rho^*(H)$.

Lemma 2.2. *Let $S \subset H$ be a submonoid.*

1. $\min \Delta_H(S) = \gcd \Delta_H(S)$.
2. *If H is commutative, then* $\min \Delta(\llbracket S \rrbracket) = \min \Delta_H(\llbracket S \rrbracket) = \min \Delta_H(S)$ *whence*

$$\Delta_\rho^*(H) = \{\min \Delta(\llbracket a \rrbracket) \mid a \in H \text{ with } \rho(\mathsf L(a)) = \rho(H)\}\,. \tag{2.4}$$

3. *Let $a, b \in S$ with $\rho(\mathsf L_H(a)) = \rho(\mathsf L_H(b)) = \rho(H)$. Then $\rho(\mathsf L_H(ab)) = \rho(H)$. In particular, $\rho(\mathsf L_H(a^k)) = \rho(H)$ for every $k \in \mathbb N$ and $\rho(\llbracket a \rrbracket) = \rho(H)$.*

Proof. 1. It is sufficient to prove that $\min \Delta_H(S) \mid d'$ for every $d' \in \Delta_H(S)$. Let $d = \min \Delta_H(S)$ and assume to the contrary that there exists $d' \in \Delta_H(S)$ such that $d \nmid d'$.

We set $d_0 = \gcd(d, d')$. Then $d_0 < d$ and there exist $x, y \in \mathbb N$ such that $d_0 = xd - yd'$. Let $a_1, a_2 \in S$ such that $\{\ell_1, \ell_1 + d\} \subset \mathsf L_H(a_1)$ and $\{\ell_2 - d', \ell_2\} \subset \mathsf L_H(a_2)$. Thus $\{x\ell_1, x\ell_1 + d, \ldots, x\ell_1 + xd\} \subset \mathsf L_H(a_1^x)$ and $\{y\ell_2 - yd', y\ell_2 - (y-1)d', \ldots, y\ell_2\} \subset \mathsf L_H(a_2^y)$. Therefore $\{x\ell_1 + y\ell_2, x\ell_1 + y\ell_2 + xd - yd'\} \subset \mathsf L_H(a_1^x a_2^y)$ which implies that $d \le xd - yd' = d_0$, a contradiction.

2. Suppose that H is commutative. Since $S \subset \llbracket S \rrbracket$ and $\llbracket S \rrbracket \subset H$ is divisor-closed, it follows that

$$\min \Delta(\llbracket S \rrbracket) = \min \Delta_H(\llbracket S \rrbracket) \le \min \Delta_H(S)\,.$$

To verify the reverse inequality, let $b \in \llbracket S \rrbracket$ with $\min \Delta(\mathsf L_H(b)) = \min \Delta(\llbracket S \rrbracket$. There is a $c \in H$ such that $bc \in S$. Since $\mathsf L_H(b) + \mathsf L_H(c) \subset \mathsf L_H(bc)$, we infer that

$$\min \Delta_H(S) \le \min \Delta(\mathsf L_H(bc)) \le \min \Delta(\mathsf L_H(b)) = \min \Delta(\llbracket S \rrbracket)\,.$$

In particular, if $S = [a]$, then $\min \Delta(\llbracket a \rrbracket) = \min \Delta_H([a])$ and hence the equation for $\Delta_\rho^*(H)$ follows.

3. Since $\mathsf L(a) + \mathsf L(b) \subset \mathsf L(ab)$, it follows that

$$\min \mathsf L(ab) \le \min \mathsf L(a) + \min \mathsf L(b) \le \max \mathsf L(a) + \max \mathsf L(b) \le \max \mathsf L(ab)\,,$$

and hence

$$\rho(H) \ge \rho(\mathsf L(ab)) = \frac{\max \mathsf L(ab)}{\min \mathsf L(ab)} \ge \frac{\max \mathsf L(a) + \max \mathsf L(b)}{\min \mathsf L(a) + \min \mathsf L(b)} \ge \min \left\{ \frac{\max \mathsf L(a)}{\min \mathsf L(a)}, \frac{\max \mathsf L(b)}{\min \mathsf L(b)} \right\} = \rho(H) \,.$$

The in particular statement follows by induction on k. □

We continue with a simple observation on the structure of the sets L_k – popping up in the definition of $\Delta_\rho(H)$ – for all monoids H under consideration. To do so, we need a further definition. Let $d \in \mathbb N$, $M \in \mathbb N_0$, and $\{0, d\} \subset \mathcal D \subset [0, d]$. A subset $L \subset \mathbb Z$ is called an *almost arithmetical multiprogression* (AAMP for short) with *difference* d, *period* $\mathcal D$, and *bound* M, if

$$L = y + (L' \cup L^* \cup L'') \, \subset \, y + \mathcal D + d\mathbb Z$$

where $y \in \mathbb Z$ is a shift parameter,

- L^* is finite nonempty with $\min L^* = 0$ and $L^* = (\mathcal D + d\mathbb Z) \cap [0, \max L^*]$, and
- $L' \subset [-M, -1]$ and $L'' \subset \max L^* + [1, M]$.

The following characterization of $\Delta_\rho(H)$ follows from the very definitions.

Lemma 2.3. *Let H be a monoid with accepted elasticity and with finite non-empty set of distances, and let $M \in \mathbb N$. Suppose that every $L \in \mathcal L(H)$ is an* AAMP *with some difference $d \in \Delta(H)$ and bound M. Then $\Delta_\rho(H)$ is the set of all $d \in \mathbb N$ with the following property: for every $k \in \mathbb N$ there is some $a_k \in H$ such that $\rho(\mathsf L(a_k)) = \rho(H)$ and*

$$\mathsf L(a_k) = y + (L' \cup \{0, d, \ldots, \ell d\} \cup L'') \subset y + d\mathbb Z$$

where $y \in \mathbb Z$, $\ell \ge k$, $L' \subset [-M, -1]$, and $L'' \subset \ell d + [1, M]$.

The assumption in Lemma 2.3, that all sets of lengths are AAMPs with global bounds, is a well-studied property in factorization theory. It holds true, among others, for transfer Krull monoids of finite type (studied in Section 3) and for weakly Krull monoids (as studied in Theorem 4.4). We refer to [17, Chapter 4.7] for a survey on settings where sets of lengths are AAMPs and also to [18]. Thus, under this assumption, the above lemma shows that the sets L_k (in Definition 2.1.2 of $\Delta_\rho(H)$) have globally bounded beginning and end parts L' and L'', and the goal is to study the set of possible distances in the middle part which can get arbitrarily long.

Lemma 2.4. *Let H be a monoid with accepted elasticity.*

1. *If $\rho(H) > 1$, then $\emptyset \ne \Delta_\rho^*(H) \subset \Delta_\rho(H)$ and $\min \Delta_\rho^*(H) = \min \Delta_\rho(H)$. In particular, if $\rho(H) > 1$ and $|\Delta(H)| = 1$, then $\Delta_\rho^*(H) = \Delta_\rho(H) = \Delta(H)$.*
2. *If $S \subset H$ is a divisor-closed submonoid with $\rho(S) = \rho(H)$, then $\Delta_\rho(S) \subset \Delta_\rho(H)$.*
3. *If H is commutative and cancellative with finitely many atoms up to associates, then $\Delta_\rho(H) \subset \{d \in \mathbb N \mid d$ divides some $d' \in \Delta_\rho^*(H)\}$. In particular, $\max \Delta_\rho(H) = \max \Delta_\rho^*(H)$.*
4. *$\Delta_\rho(H) = \emptyset$ if and only if $\Delta_1(H) = \emptyset$ if and only if $\Delta(H) = \emptyset$ if and only if $\rho(H) = 1$.*

Proof. 1. Suppose that $\rho(H) > 1$. Then, by definition, there is an $a \in H$ with $\rho(\mathsf L(a)) = \rho(H) > 1$ whence $\Delta_H([a]) \ne \emptyset$ and thus $\Delta_\rho^*(H) \ne \emptyset$. To verify that $\Delta_\rho^*(H) \subset \Delta_\rho(H)$, we set $d = \min \Delta_H([a])$. Then there is an $\ell \in \mathbb N$ such that $d \in \Delta(\mathsf L(a^\ell))$ and thus for every $k \in \mathbb N$ the set $\mathsf L(a^{k\ell})$ contains an arithmetical progression with difference d and length at least k. Since $\min \Delta_H([a]) = \gcd \Delta_H([a])$ by Lemma 2.2.2, $\mathsf L(a^{k\ell})$ is an AAP with difference d and length at least k for every $k \in \mathbb N$. By Lemma 2.2.3, we have $\rho(\mathsf L_H(a^{k\ell})) = \rho(H)$ and thus $d \in \Delta_\rho(H)$.

Since $\Delta_\rho^*(H) \subset \Delta_\rho(H)$, it follows that $\min \Delta_\rho(H) \le \min \Delta_\rho^*(H)$. To verify the reverse inequality, let $d \in \Delta_\rho(H)$ be given. Then there is an $a \in H$ such that $\mathsf L(a)$ is an AAP with difference d, length at least 1, and $\rho(\mathsf L(a)) = \rho(H)$. Thus $\min \Delta_H([a]) \in \Delta_\rho^*(H)$ by definition, and clearly we have $\min \Delta_H([a]) \le \min \Delta(\mathsf L(a)) = d$.

If $\rho(H) > 1$ and $|\Delta(H)| = 1$, then the inclusions given in (2.2) imply that $\Delta^*_\rho(H) = \Delta_\rho(H) = \Delta(H)$.

2. Suppose that $S \subset H$ is divisor-closed with $\rho(S) = \rho(H)$. Then for every $a \in S$, we have $\mathsf L_S(a) = \mathsf L_H(a)$, and hence $\mathcal L(S) \subset \mathcal L(H)$. If $d \in \Delta_\rho(S)$, then by definition, for every $k \in \mathbb N$, there is some $L_k \in \mathcal L(S) \subset \mathcal L(H)$ which is an AAP with difference d, length at least k, and with $\rho(L_k) = \rho(S) = \rho(H)$, and thus $d \in \Delta_\rho(H)$.

3. Clearly, the in particular statement follows from the asserted inclusion and from the fact that $\Delta^*_\rho(H) \subset \Delta_\rho(H)$ as shown in 1. We use several times the fact that finitely generated commutative monoids are locally tame and have accepted elasticity ([17, Theorem 3.1.4]).

Without restriction we may suppose that H is reduced, and we set $\mathcal A(H) = \{u_1, \ldots, u_t\}$ with $t \in \mathbb N$. Let $d \in \Delta_\rho(H)$ be given. Then for every $k \in \mathbb N$, there is a $b_k \in H$ such that $\rho\big(\mathsf L(b_k)\big) = \rho(H)$ and $\mathsf L(b_k)$ is an AAP with difference d and length $\ell_k \ge k$. Since $\mathcal A(H)$ is finite, there are a nonempty subset $A \subset \mathcal A(H)$, say $A = \{u_1, \ldots, u_s\}$ with $s \in [1,t]$, a constant $M_1 \in \mathbb N_0$, and a subsequence $(b_{m_k})_{k\ge1}$ of $(b_k)_{k\ge1}$, say $b_{m_k} = c_k$ for all $k \in \mathbb N$, such that, again for all $k \in \mathbb N$,

$$c_k = \prod_{i=1}^t u_i^{m_{k,i}} \quad \text{where} \quad m_{k,i} \ge k \text{ for } i \in [1,s] \quad \text{and} \quad m_{k,i} \le M_1 \text{ for } i \in [s+1,t]$$

By Theorem [17, Theorem 4.3.6] (applied to the monoid $[\![u_1 \cdot \ldots \cdot u_s]\!]$), $L_k = \mathsf L(\prod_{i=1}^s u_i^{m_{k,i}})$ is an AAP with difference $d' = \min \Delta([\![u_1 \cdot \ldots \cdot u_s]\!])$ for every $k \in \mathbb N$. Since H is locally tame, [17, Proposition 4.3.4] implies that there is a constant $M_2 \in \mathbb N_0$ such that for every $k \in \mathbb N$

$$\max \mathsf L(c_k) \le \max L_k + M_2 \quad \text{and} \quad \min \mathsf L(c_k) \ge \min L_k - M_2 \,.$$

Since for every $k \in \mathbb N$ there is a $y_k \in \mathbb N$ such that $y_k + L_k \subset \mathsf L(c_k)$, we infer that d divides d'. Being a divisor-closed submonoid of a finitely generated monoid, the monoid $[\![u_1 \cdot \ldots \cdot u_s]\!]$ is finitely generated by [17, Proposition 2.7.5]. Thus there is an $a \in [\![u_1 \cdot \ldots \cdot u_s]\!]$ such that $\rho(\mathsf L(a)) = \rho([\![u_1 \cdot \ldots \cdot u_s]\!])$. Since d divides $d' = \min \Delta([\![u_1 \cdot \ldots \cdot u_s]\!])$ and d' divides $\min \Delta([\![a]\!])$, it follows that d divides $\min \Delta([\![a]\!])$.

Next we verify that $\rho([\![u_1 \cdot \ldots \cdot u_s]\!]) = \rho(H)$ from which it follows that $\min \Delta([\![a]\!]) \in \Delta^*_\rho(H)$ by Lemma 2.2.2. For $k \in \mathbb N$, we have

$$\rho(H) = \frac{\max \mathsf L(c_k)}{\min \mathsf L(c_k)} \le \frac{\max L_k + M_2}{\min L_k - M_2} \quad \text{and} \quad \frac{\max L_k}{\min L_k} \le \rho([\![u_1 \cdot \ldots \cdot u_s]\!]) \le \rho(H)\,.$$

If k tends to infinity, then $(\max L_k + M_2)/(\min L_k - M_2)$ tends to $\max \mathsf L(c_k)/\min \mathsf L(c_k)$ which implies that $\rho([\![u_1 \cdot \ldots \cdot u_s]\!]) = \rho(H)$.

4. This follows from 1. and from the basic relation given in (2.2). □

Lemma 2.5. *Let H be a monoid with accepted elasticity. Then for every nonempty subset $\Delta \subset \Delta_\rho(H)$ there is a $d \in \Delta_\rho(H)$ such that $d \le \gcd \Delta$.*

Proof. Let $\Delta = \{d_1, \ldots, d_n\} \subset \Delta_\rho(H)$ be a nonempty subset. For every $i \in [1,n]$ and every $k \in \mathbb N$ there is an $a_{i,k} \in H$ such that $\mathsf L(a_{i,k})$ is an AAP with difference d_i, length at least k, and with $\rho(\mathsf L(a_{i,k})) = \rho(H)$. By Lemma 2.2.3, $\mathsf L(a_{1,k} \cdot \ldots \cdot a_{n,k})$ has elasticity $\rho(H)$ for all $k \in \mathbb N$, and thus $d = \min \Delta_H([a_{1,k} \cdot \ldots \cdot a_{n,k}]) \in \Delta^*_\rho(H) \subset \Delta_\rho(H)$. If k is sufficiently large, then $\gcd(d_1, \ldots, d_n)$ occurs as a distance of the sumset $\mathsf L(a_{1,k}) + \ldots + \mathsf L(a_{n,k})$. Since the sumset

$$\mathsf L(a_{1,k}) + \ldots + \mathsf L(a_{n,k}) \subset \mathsf L(a_{1,k} \cdot \ldots \cdot a_{n,k})$$

and $d = \gcd \Delta_H([a_{1,k} \cdot \ldots \cdot a_{n,k}])$ by Lemma 2.2.1, d divides any distance of $\Delta\big(\mathsf L(a_{1,k} \cdot \ldots \cdot a_{n,k})\big)$ whence it divides $\gcd(d_1, \ldots, d_n)$. □

Lemma 2.6. *Let $H = H_1 \times \ldots \times H_n$ where $n \in \mathbb N$ and $H_1, \ldots, H_n$ are atomic unit-cancellative monoids.*

1. *Then* $\rho(H) = \sup\{\rho(H_1), \ldots, \rho(H_n)\}$*, and* H *has accepted elasticity if and only if there is some* $i \in [1, s]$ *such that* H_i *has accepted elasticity* $\rho(H_i) = \rho(H)$*.*
2. *Let* $s \in [1, n]$ *and suppose that* H_i *has accepted elasticity* $\rho(H_i) = \rho(H)$ *for all* $i \in [1, s]$*, and that* H_i *either does not have accepted elasticity or* $\rho(H_i) < \rho(H)$ *for all* $i \in [s + 1, t]$*. We set*

$$\Delta' = \big\{ \gcd\{d_i \mid i \in I\} \mid d_i \in \Delta_\rho(H_i) \text{ for all } i \in I, \emptyset \ne I \subset [1, s]\big\} \quad \text{and}$$
$$\Delta'' = \big\{ \gcd\{d_i \mid i \in I\} \mid d_i \in \Delta^*_\rho(H_i) \text{ for all } i \in I, \emptyset \ne I \subset [1, s]\big\}.$$

Then $\Delta' \subset \Delta_\rho(H)$*,* $\Delta'' \subset \Delta^*_\rho(H)$*, and if* $|\Delta(H_i)| = 1$ *for all* $i \in [1, s]$*, then* $\Delta' = \Delta'' = \Delta^*_\rho(H) = \Delta_\rho(H)$*.*

Proof. 1. The formula for $\rho(H)$ follows from [17, Proposition 1.4.5], where a proof is given for cancellative monoids but the proof of the general case runs along the same lines. The formula for $\rho(H)$ immediately implies the second assertion.

2.(i) First we show that $\Delta' \subset \Delta_\rho(H)$. Let $\emptyset \ne I \subset [1, s]$, say $I = [1, r]$, and choose $d_i \in \Delta_\rho(H_i)$ for every $i \in [1, r]$. For each $i \in [1, r]$ and every $\ell \in \mathbb{N}$ there is an $a_{i,\ell} \in H_i$ such that $\mathsf{L}(a_{i,\ell})$ is an AAP with difference d_i, length at least 2ℓ, and with $\rho(\mathsf{L}(a_{i,\ell})) = \rho(H)$. Then $\rho\big(\mathsf{L}(a_{1,\ell} \cdot \ldots \cdot a_{r,\ell})\big) = \rho(H)$ by Lemma 2.2.3. Thus, for all sufficiently large ℓ, the sumset $\mathsf{L}(a_{1,\ell}) + \ldots + \mathsf{L}(a_{r,\ell}) = \mathsf{L}(a_{1,\ell} \cdot \ldots \cdot a_{r,\ell})$ is an AAP with difference $\gcd(d_1, \ldots, d_r)$ and length at least ℓ.

2.(ii) Second we show that $\Delta'' \subset \Delta^*_\rho(H)$. Let $\emptyset \ne I \subset [1, s]$, say $I = [1, r]$, and choose $d_i \in \Delta^*_\rho(H_i)$ for every $i \in [1, r]$. Thus there are $a_i \in H_i$ such that $\rho(\mathsf{L}(a_i)) = \rho(H)$ and $\min \Delta_{H_i}([a_i]) = \min \Delta_H([a_i]) = d_i$ for all $i \in [1, r]$. Therefore, again for all $i \in [1, r]$, there is an $\ell_i \in \mathbb{N}$ such that $d_i \in \Delta(\mathsf{L}(a_i^{\ell_i}))$ and thus, for every $k \in \mathbb{N}$, $\mathsf{L}(a_i^{2k\ell_i})$ contains an arithmetical progression with difference d_i and length at least $2k$. Setting $\ell = \max(\ell_1, \ldots, \ell_r)$ we infer that

$$\mathsf{L}((a_1 \cdot \ldots \cdot a_r)^{2k\ell}) = \mathsf{L}(a_1^{2k\ell}) + \ldots + \mathsf{L}(a_r^{2k\ell})$$

is an AAP with difference $\gcd(d_1, \ldots, d_r)$ and length at least k for all sufficiently large k. Thus $\min \Delta_H([a_1 \cdot \ldots \cdot a_r]) = \gcd(d_1, \ldots, d_r)$. Since $\rho(\mathsf{L}(a_1 \cdot \ldots \cdot a_r)) = \rho(H)$ by Lemma 2.2.3, it follows that $\gcd(d_1, \ldots, d_r) = \min \Delta_H([a_1 \cdot \ldots \cdot a_r]) \in \Delta^*_\rho(H)$.

2.(iii) Now suppose that $\Delta(H_i) = \{d_i\}$ for all $i \in [1, s]$. Then $\Delta^*_\rho(H_i) = \Delta_\rho(H_i) = \Delta(H_i)$ by Lemma 2.4.1 and hence $\Delta' = \Delta''$. By 2.(i) and 2.(ii) it remains to show that $\Delta_\rho(H) \subset \Delta'$. Then all four sets are equal as asserted.

Let $d \in \Delta_\rho(H)$ and let $k \in \mathbb{N}$ be sufficiently large. Then there are $a_{1,k} \in H_1, \ldots, a_{s,k} \in H_s$ such that $\mathsf{L}(a_{1,k} \cdot \ldots \cdot a_{s,k})$ is an AAP with difference d, elasticity $\rho(H)$, and length at least k. Since $\Delta(H_i) = \{d_i\}$ for all $i \in [1, s]$,

$$\mathsf{L}(a_{1,k} \cdot \ldots \cdot a_{s,k}) = \mathsf{L}(a_{1,k}) + \ldots + \mathsf{L}(a_{s,k})$$

is a sumset of arithmetical progressions with differences $d_1, \ldots, d_s$. Since $\mathsf{L}(a_{1,k} \cdot \ldots \cdot a_{s,k})$ is an AAP with difference d and length at least k with k being sufficiently large, it follows that $d = \gcd(d_1, \ldots, d_s)$ and hence $d \in \Delta'$. □

3. Transfer Krull monoids

An atomic unit-cancellative monoid H is said to be a *transfer Krull monoid* if one of the following two equivalent properties is satisfied:

(a) There is a commutative Krull monoid B and a weak transfer homomorphism $\theta\colon H \to B$.
(b) There is an abelian group G, a subset $G_0 \subset G$, and a weak transfer homomorphism $\theta\colon H \to \mathcal{B}(G_0)$.

In case (b) we say that H is a transfer Krull monoid over G_0, and if G_0 is finite, then H is said to be a transfer Krull monoid of finite type. We do not repeat the technical definition of weak transfer homomorphisms (introduced by Baeth and Smertnig in [3]) because we use only that they preserve sets of lengths. Therefore $\mathcal{L}(H) = \mathcal{L}(G_0)$ ([15, Lemma 4.2]) which, by definition, implies that

$$\Delta(H) = \Delta(G_0),\ \Delta_\rho(H) = \Delta_\rho(G_0),\ \rho(H) = \rho(G_0)\,, \tag{3.1}$$

and H has accepted elasticity if and only if $\mathcal{B}(G_0)$ has accepted elasticity. Note that, as with other invariants, we use the abbreviations

$$\Delta_1(G_0) := \Delta_1\big(\mathcal{B}(G_0)\big), \quad \Delta_\rho^*(G_0) := \Delta_\rho^*\big(\mathcal{B}(G_0)\big), \quad \text{and} \quad \Delta_\rho(G_0) := \Delta_\rho\big(\mathcal{B}(G_0)\big)\,.$$

Every commutative Krull monoid (and thus every commutative Krull domain) with class group G is a transfer Krull monoid over the subset $G_0 \subset G$ containing prime divisors. In particular, if the class group G is finite and every class contains a prime divisor (which holds true for holomorphy rings in global fields), then it is a transfer Krull monoid over G. Deep results, due to D. Smertnig, reveal large classes of bounded HNP (hereditary noetherian prime) rings to be transfer Krull ([36, 3, 35]). To mention one of these results in detail, let $\mathcal{O}$ be a ring of integers of an algebraic number field K, A a central simple algebra over K, and R a classical maximal $\mathcal{O}$-order of A. Then the monoid of cancellative elements of R is transfer Krull if and only if every stably free left R-ideal is free, and if this holds, then it is a tranfer Krull monoid over a finite abelian group (namely a ray class group of $\mathcal{O}$). We refer to [15] for a detailed discussion of commutative Krull monoids with finite class group and of further transfer Krull monoids.

Let H be a transfer Krull monoid over a finite abelian group G. The system $\mathcal{L}(H) = \mathcal{L}(G)$, together with all parameters controlling it, is a central object of interest in factorization theory (see [34] for a survey). By (2.2) and Lemma 2.4.1, we have

$$\Delta_\rho^*(G) \subset \Delta_\rho(G) \subset \Delta_1(G) \subset \Delta(G)\,.$$

The set $\Delta(G)$ is an interval by [22], but $\Delta_1(G)$ is far from being an interval ([32]). A characterization when $\Delta_1(G)$ is an interval can be found in [37]. We have $\max \Delta_1(G) = \max\{\mathsf{r}(G) - 1 \exp(G) - 2\}$ (for $|G| \ge 3$, by [24]). This section will reveal that $\Delta_\rho(G)$ is quite different from $\Delta_1(G)$.

We start with a result for transfer Krull monoids over arbitrary finite subsets. It shows that in finitely generated commutative Krull monoids H with finite class group (and without restriction on the classes containing prime divisors) a large variety of finite sets can be realized as $\Delta_\rho(H)$ sets (Lemma 2.5 shows that not every finite set can be realized as a $\Delta_\rho(\cdot)$ set of some monoid; see also Lemmas 2.6 and 4.3). In contrast to this we will see that the set $\Delta_\rho(H)$ is extremely restricted if the set of classes containing prime divisors is very large.

Theorem 3.1.

1. *Let H be a transfer Krull monoid over a finite subset G_0. Then H has accepted elasticity $\rho(H) = \rho(G_0) \le \mathsf{D}(G_0)/2$ and equality holds if $G_0 = -G_0$.*
2. *For every finite set $\Delta = \{d_1, \ldots, d_n\} \subset \mathbb{N}$ there exists a finitely generated commutative Krull monoid H with finite class group such that $\{\gcd\{d_i \mid i \in I\} \mid \emptyset \ne I \subset [1,n]\} = \Delta_\rho^*(H) = \Delta_\rho(H)$.*
3. *If H be a transfer Krull monoid over a subset G_0 of a finite abelian group G with $\rho(H) = \mathsf{D}(G)/2$, then $\langle G_0 \rangle = G$ and $\Delta_\rho(H) \subset \Delta_\rho(G)$.*

Proof. 1. By (3.1), we have $\mathcal{L}(H) = \mathcal{L}(G_0)$ and hence $\rho(H) = \rho(G_0)$. Since the set G_0 is finite, the monoid $\mathcal{B}(G_0)$ is finitely generated whence the elasticity $\rho(G_0)$ is accepted ([17, Theorems 3.1.4 and 3.4.2]). The statements on $\rho(G_0)$ follow from [17, Theorem 3.4.11].

2. Let $\Delta = \{d_1, \ldots, d_n\} \subset \mathbb{N}$ be a finite set. We start with the following assertion.

A. For every $i \in [1,n]$, there is a finite abelian group G_i and a subset $G_i' \subset G_i$ such that $\Delta_\rho(G_i') = \Delta(G_i') = \{d_i\}$ and $\rho(G_i') = 2$.

Proof of **A**. We do the construction for a given $d \in \mathbb{N}$ and omit all indices. If $d = 1$, then $G = C_8 = \{0, g, \ldots, 7g\}$ and $G' = \{g, 3g\}$ have the required properties. Suppose that $d \ge 2$. Consider a finite abelian group G, independent elements $e_1, \ldots, e_{d-1} \in G$ with $\mathrm{ord}(e_1) = \ldots = \mathrm{ord}(e_{d-1}) = 2d$, and set $e_0 = -(e_1 + \ldots + e_{d-1})$. It is easy to check that $G' = \{e_0, e_1, \ldots, e_{d-1}\}$ satisfies $\rho(G') = 2$ and $\Delta(G') = \{d\}$ (for details of a more general construction see [17, Proposition 4.1.2]). □[Proof of **A**]

We set

$$G_0 = \biguplus_{i=1}^{n} G_i' \ \subset \ G = G_1 \oplus \ldots \oplus G_n \quad \text{and} \quad H = \mathcal{B}(G_0)\,.$$

Then $H = \mathcal{B}(G_1') \times \ldots \times \mathcal{B}(G_n')$ is a finitely generated commutative Krull monoid with finite class group. By Lemma 2.6.1, H has accepted elasticity $\rho(H) = 2$ and

$$\big\{ \gcd\{d_i \mid I \subset [1,n]\} \mid \emptyset \ne I \subset [1,n] \big\} = \Delta_\rho^*(H) = \Delta_\rho(H)\,.$$

3. Let H be a transfer Krull monoid over G_0 such that $\rho(H) = \mathsf{D}(G)/2$. Then 1. shows that

$$\mathsf{D}(G)/2 = \rho(H) \le \mathsf{D}(G_0)/2 \le \mathsf{D}(G)/2\,.$$

Thus

$$\mathsf{D}(G) = \mathsf{D}(G_0) \le \mathsf{D}(\langle G_0 \rangle) \le \mathsf{D}(G)\,,$$

and since proper subgroups of G have a strictly smaller Davenport constant ([17, Proposition 5.1.11]), it follows that $\langle G_0 \rangle = G$.

Since $\rho(H) = \rho(G_0)$ and $\rho(G) = \mathsf{D}(G)/2$ by 1., we obtain that $\rho(G_0) = \rho(G)$. Since $\Delta_\rho(H) = \Delta_\rho(G_0)$ and $\mathcal{B}(G_0) \subset \mathcal{B}(G)$ is a divisor-closed submonoid, the assertion follows from Lemma 2.4.2. □

Let all notation be as in Theorem 3.1.3. Since $\Delta_\rho(H) \ne \emptyset$ and $\Delta_\rho(G)$ will turn out to be small (Conjecture 3.20), we have $\Delta_\rho(H) = \Delta_\rho(G)$ in many situations (as it holds true in the case $G_0 = G$).

In the remainder of this section we study $\Delta_\rho(G)$ for finite abelian groups G. Suppose that

$$G \cong C_{n_1} \oplus \ldots \oplus C_{n_r} \quad \text{and set} \quad \mathsf{D}^*(G) = 1 + \sum_{i=1}^{r} (n_i - 1)\,, \tag{3.2}$$

where $1 < n_1 \mid \ldots \mid n_r$, $n_r = \exp(G)$ is the exponent of G, and $r = \mathsf{r}(G)$ is the rank of G. Thus $\mathsf{r}(G) = \max\{\mathsf{r}_p(G) \mid p \in \mathbb{P}\}$ is the maximum of all p-ranks $\mathsf{r}_p(G)$ over all primes $p \in \mathbb{P}$.

The next lemma, Lemma 3.2, reveals that the study of $\Delta_\rho(G)$ needs information on the Davenport constant $\mathsf{D}(G)$ as well as (at least some basic) information on the structure of minimal zero-sum sequences having length $\mathsf{D}(G)$. Although studied since the 1960s, the precise value of the Davenport constant is known only in a very limited number of cases. Clearly, we have $\mathsf{D}^*(G) \le \mathsf{D}(G)$ and since the 1960s it is known that equality holds if $\mathsf{r}(G) \le 2$ or if G is a p-group. Further classes of groups have been found where equality holds and also where it does not hold, but a good understanding of this phenomenon is still missing. Even less is known on the inverse problem, namely on the structure of minimal zero-sum sequences having length $\mathsf{D}(G)$. The structure of such sequences is clear for cyclic groups and for elementary 2-groups, and recently the structure was determined for rank two groups. For general groups, even harmless looking questions (such as whether each minimal zero-sum sequence of length $\mathsf{D}(G)$ does contain an element of order $\exp(G)$) are open. In this section we study $\Delta_\rho(G)$ for all classes of groups where at least some information on the inverse problem is available.

Recall that $\Delta(G) = \emptyset$ if and only if $|G| \le 2$ whence we will always assume that $|G| \ge 3$.

Lemma 3.2. *Let G be a finite abelian group with $|G| \ge 3$.*

1. *For $A \in \mathcal{B}(G)$ the following statements are equivalent*:
   (a) $\rho\big(\mathsf{L}(A)\big) = \mathsf{D}(G)/2$.
   (b) *There are $k, \ell \in \mathbb{N}$ and $U_1, \ldots, U_k, V_1, \ldots, V_\ell \in \mathcal{A}(G)$ with $|U_1| = \ldots = |U_k| = \mathsf{D}(G)$, $|V_1| = \ldots = |V_\ell| = 2$ such that $A = U_1 \cdot \ldots \cdot U_k = V_1 \cdot \ldots \cdot V_\ell$.*

2. *For a subset $G_0 \subset G$ the following statements are equivalent*:
   (a) $G_0 = \text{supp}(A)$ *for some* $A \in \mathcal{B}(G)$ *with* $\rho(\mathsf{L}(A)) = \mathsf{D}(G)/2$.
   (b) $G_0 = -G_0$ *and for every* $g \in G_0$ *there is some* $A \in \mathcal{A}(G_0)$ *with* $g \mid A$ *and* $|A| = \mathsf{D}(G)$.
3. $\Delta_\rho^*(G) = \{\min \Delta(G_0) \mid G_0 = \text{supp}(A) \text{ for some } A \in \mathcal{B}(G) \text{ with } \rho(\mathsf{L}(A)) = \mathsf{D}(G)/2\}$.

Comment on 1. If $U_1, \ldots, U_m \in \mathcal{A}(G)$ with $|U_1| = \ldots = |U_m| = \mathsf{D}(G)$, then obviously we obtain an equation of the form $U_1(-U_1) \cdot \ldots \cdot U_m(-U_m) = V_1 \cdot \ldots \cdot V_{m\mathsf{D}(G)}$ with $|V_i| = 2$ for all $i \in [1, m\mathsf{D}(G)]$. But there are also equations $U_1 \cdot \ldots \cdot U_k = V_1 \cdot \ldots \cdot V_\ell$ with all properties as in 1.(b) and with k odd ([12]).

Proof. 1. (a) $\Rightarrow$ (b) We set $L = \mathsf{L}(A)$ and suppose that $\rho(L) = \mathsf{D}(G)/2$. If $A = 0^m C$, with $m \in \mathbb{N}_0$ and $C \in \mathcal{B}(G \setminus \{0\})$, then

$$\frac{\mathsf{D}(G)}{2} = \frac{\max L}{\min L} = \frac{m + \max \mathsf{L}(C)}{m + \min \mathsf{L}(C)} \le \frac{\max \mathsf{L}(C)}{\min \mathsf{L}(C)} \le \frac{\mathsf{D}(G)}{2}\,,$$

whence $m = 0$. Suppose that

$$U_1 \cdot \ldots \cdot U_k = A = V_1 \cdot \ldots \cdot V_\ell\,,$$

with $k = \min \mathsf{L}(A)$, $\ell = \max \mathsf{L}(A)$, and $U_1, \ldots, U_k, V_1, \ldots, V_\ell \in \mathcal{A}(G)$. Then $\rho(L) = \ell/k = \mathsf{D}(G)/2$ and

$$2\ell \le \sum_{i=1}^{\ell} |V_i| = |A| = \sum_{i=1}^{k} |U_i| \le k\mathsf{D}(G)\,.$$

This implies that $|A| = 2\ell = k\mathsf{D}(G)$, $|V_1| = \ldots = |V_\ell| = 2$, and $|U_1| = \ldots = |U_k| = \mathsf{D}(G)$.

(b) $\Rightarrow$ (a) Suppose that $A = U_1 \cdot \ldots \cdot U_k = V_1 \cdot \ldots \cdot V_\ell$ where $U_1, \ldots, U_k, V_1, \ldots, V_\ell$ are as in (b). Then we infer that

$$\min \mathsf{L}(A)\mathsf{D}(G) \le k\mathsf{D}(G) = |A| = 2\ell \le 2 \max \mathsf{L}(A)$$

and hence

$$\frac{\mathsf{D}(G)}{2} \le \frac{\max \mathsf{L}(A)}{\min \mathsf{L}(A)} = \rho\big(\mathsf{L}(A)\big) \le \frac{\mathsf{D}(G)}{2}\,.$$

2.(a) $\Rightarrow$ (b) This follows from 1.

(b) $\Rightarrow$ (a) We set $G_0 = \{g_1, -g_1, \ldots, g_k, -g_k\}$. For every $i \in [1, k]$, let $A_i \in \mathcal{A}(G_0)$ with $g_i \mid A_i$ and $|A_i| = \mathsf{D}(G)$, and set $A = \prod_{i=1}^k (-A_i)A_i$. Then $\text{supp}(A) = G_0$ and $\rho\big(\mathsf{L}(A)\big) = \mathsf{D}(G)/2$.

3. Since for every $A \in \mathcal{B}(G)$ we have $[\![A]\!] = \mathcal{B}(\text{supp}(A))$, the assertion follows from 2. $\square$

Corollary 3.3. *Let G be a finite abelian group with $|G| \ge 3$.*
1. $\Delta_\rho^*(G) \subset \Delta_\rho(G) \subset \{d \in \mathbb{N} \mid d$ *divides some* $d' \in \Delta_\rho^*(G)\}$.
2. $\max \Delta_\rho(G) = \max \Delta_\rho^*(G) = \max\{\min \Delta(G_0) \mid G_0 = \text{supp}\big((-U)U\big), U \in \mathcal{A}(G_0)$ *with* $|U| = \mathsf{D}(G)\}$.

Proof. 1. Since $\mathcal{B}(G)$ is finitely generated, this follows from Lemma 2.4.

2. The first equality follows from 1. Then Lemma 3.2.3 implies that

$$\max \Delta_\rho^*(G) = \max\{\min \Delta(G_0) \mid G_0 = \text{supp}(A) \text{ for some } A \in \mathcal{B}(G) \text{ with } \rho(\mathsf{L}(A)) = \mathsf{D}(G)/2\}\,.$$

Let $A \in \mathcal{B}(G)$ with $G_0 = \text{supp}(A)$ and $\rho(\mathsf{L}(A)) = \mathsf{D}(G)/2$. Then, by Lemma 3.2, $G_0 = -G_0$ and $A = U_1 \cdot \ldots \cdot U_k$ with $U_1, \ldots, U_k \in \mathcal{A}(G)$ and $|U_1| = \ldots = |U_k| = \mathsf{D}(G)$. Then $G_1 = \text{supp}\big((-U_1)U_1\big) \subset G_0$ and $\min \Delta(G_0) \le \min \Delta(G_1)$. Thus the assertion follows. $\square$

Let G be a finite abelian group and let $g \in G$ with $\text{ord}(g) = n \ge 2$. For every sequence $S = (n_1 g) \cdot \ldots \cdot (n_\ell g) \in \mathcal{F}(\langle g \rangle)$, where $\ell \in \mathbb{N}_0$ and $n_1, \ldots, n_\ell \in [1, n]$, we define its g-norm

$$\|S\|_g = \frac{n_1 + \ldots + n_\ell}{n}\,.$$

Note that, $\sigma(S) = 0$ implies that $n_1 + \ldots + n_\ell \equiv 0 \mod n$ whence $\|S\|_g \in \mathbb{N}_0$.

Lemma 3.4. *Let G be a finite abelian group with $|G| \ge 3$ and $G_0 \subset G$ be a subset.*

1. *If $-G_0 = G_0$, then $\min \Delta(G_0)$ divides $\gcd\{|U|-2 \mid U \in \mathcal{A}(G_0)\}$.*
2. *If $r \ge 2$, $(e_1, \ldots, e_r)$ independent, $\operatorname{ord}(e_i) = n_i$ for all $i \in [1,r]$ where $n_1 \mid \ldots \mid n_r$, $n_r > 2$, $e_0 = e_1 + \ldots + e_r$, and $G_0 = \{e_1, -e_1, \ldots, e_r, -e_r, e_0, -e_0\}$, then $\min \Delta(G_0) = 1$.*
3. *If $\langle G_0 \rangle = \langle g \rangle$ for some $g \in G_0$ and $\Delta(G_0) \ne \emptyset$, then $\min \Delta(G_0) = \gcd\big\{\|V\|_g - 1 \ \big|\ V \in \mathcal{A}(G_0)\big\}$.*

Proof. 1. If $U = g_1 \cdot \ldots \cdot g_\ell \in \mathcal{A}(G_0)$, then $(-U)U = \prod_{i=1}^{\ell} \big((-g_i)g_i\big)$ whence $\{2, \ell\} \subset \mathsf{L}\big((-U)U\big)$ and so $\gcd \Delta(G_0)$ divides $\ell - 2$.

2. Since $e_0 = e_1 + \ldots + e_r$, we have $\operatorname{ord}(e_0) = n_r > 2$. We distinguish two cases. First, suppose that $n_1 > 2$. Then

$$W = e_0^{n_r-1} e_1 \cdot \ldots \cdot e_{r-1}(-e_r)^{n_r-1} \in \mathcal{A}(G_0)$$

and $W^2 = e_0^{n_r} \cdot (-e_r)^{n_r} \cdot \big(e_0^{n_r-2} e_1^2 \cdot \ldots \cdot e_{r-1}^2 (-e_r)^{n_r-2}\big)$ is a product of three atoms whence $\min \Delta(G_0) = 1$.

Now, we suppose that $n_1 = 2$, and let $t \in [1, r-1]$ such that $n_1 = \ldots = n_t = 2$ and $n_{t+1} > 2$. Then

$$S_1 = e_0 e_1 \cdot \ldots \cdot e_t (-e_{t+1}) \cdot \ldots \cdot (-e_r) \in \mathcal{A}(G_0) \quad \text{and} \quad S_2 = e_0^{n_r-1} e_1 \cdot \ldots \cdot e_r \in \mathcal{A}(G_0)\,.$$

Then

$$S_1^2 = \big(e_0^2 (-e_{t+1})^2 \cdot \ldots \cdot (-e_r)^2\big) e_1^2 \cdot \ldots \cdot e_t^2$$

is a product of $t+1$ atoms and

$$S_2^2 = e_0^{n_r} \cdot \big(e_0^{n_r-2} e_{t+1}^2 \cdot \ldots \cdot e_r^2\big) \cdot e_1^2 \cdot \ldots \cdot e_t^2$$

is a product of $t+2$ atoms. Thus $\min \Delta(G_0) \mid \gcd(t+1-2, t+2-2) = 1$ which implies that $\min \Delta(G_0) = 1$.

3. See [17, Lemma 6.8.5]. ☐

Theorem 3.5. *Let H be a transfer Krull monoid over a finite abelian group G with $|G| \ge 3$. Then $1 \in \Delta_\rho(H)$ if and only if G is not cyclic of order $4, 6$ or 10.*

Proof. By (3.1), it is sufficient to prove the assertion for $\mathcal{B}(G)$ instead of H. We distinguish two cases.

CASE 1: $\mathsf{r}(G) \ge 2$.

By Corollary 3.3.1, it is sufficient to prove that $1 \in \Delta_\rho^*(G)$. For each prime p dividing $|G|$, we denote by G_p the Sylow p-subgroup of G. Since $\mathsf{r}(G) \ge 2$, there exists a Sylow-p subgroup G_p such that $\mathsf{r}(G_p) \ge 2$. We distinguish two subcases.

CASE 1.1: There exists a Sylow p-subgroup G_p such that $\mathsf{r}(G_p) \ge 2$ and $\exp(G_p) \ge 3$.

Then there exists a subgroup H of G with $p \nmid |H|$ such that $G \cong G_p \oplus H$ (clearly, we may have $H = \{0\}$). Let A be an atom of $\mathcal{B}(G)$ with length $|A| = \mathsf{D}(G)$. Thus for every g dividing A, there exists a unique pair (f_g, h_g) with $f_g \in G_p$ and $h_g \in H$ such that $g = f_g + h_g$. Since $\langle \operatorname{supp}(A) \rangle = G$, there must exist $g \in \operatorname{supp}(A)$ such that $\operatorname{ord}(f_g) = \exp(G_p)$. Therefore we can find $e_2, \ldots, e_{\mathsf{r}(G_p)}$ such that $G_p = \langle f_g \rangle \oplus \langle e_2 \rangle \oplus \ldots \oplus \langle e_{\mathsf{r}(G_p)} \rangle$. There are group isomorphisms

$$\phi : G \longrightarrow G \ \text{ by } \phi(f_g) = f_g + e_2,\ \phi(e_i) = e_i \text{ for each } i \in [2, \mathsf{r}(G_p)], \text{ and } \phi(h) = h \text{ for each } h \in H\,,$$

and

$$\psi : G \longrightarrow G \ \text{ by } \psi(f_g) = f_g - e_2,\ \psi(e_i) = e_i \text{ for each } i \in [2, \mathsf{r}(G_p)], \text{ and } \psi(h) = h \text{ for each } h \in H\,.$$

It follows that $\phi(A)$ and $\psi(A)$ are atoms of length $\mathsf{D}(G)$. We consider the set

$$G_0 = \operatorname{supp}\big((-A)A\,\phi((-A)A)\,\psi((-A)A)\big)\,.$$

Obviously, we have $G_0 = -G_0$ and for every $a \in G_0$ there is some $A' \in \mathcal{A}(G_0)$ with $a \mid A'$ and $|A'| = \mathsf{D}(G)$. Thus it is sufficient to prove $\min \Delta(G_0) = 1$. Since

$$\{g, -g, \phi(g), \psi(g)\} = \{g, -g, g + e_2, g - e_2\} \subset G_0$$

and

$$\mathsf{L}(g^{\mathrm{ord}(g)}(-g)^{\mathrm{ord}(g)}) = \{2, \mathrm{ord}(g)\} \quad \text{and} \quad \mathsf{L}(g^{\mathrm{ord}(g)-2}(g+e_2)(g-e_2)(-g)^{\mathrm{ord}(g)}) = \{2, \mathrm{ord}(g)-1\}\,,$$

it follows that $\min \Delta(G_0) \mid \gcd\{\mathrm{ord}(g)-2, \mathrm{ord}(g)-3\}$. Since $\mathrm{ord}(g) \ge \exp(G_p) \ge 3$, we obtain that $\min \Delta(G_0) = 1$.

CASE 1.2: There is no Sylow p-subgroup G_p such that $\mathsf{r}(G_p) \ge 2$ and $\exp(G_p) \ge 3$.

Let G_p be the Sylow p-subgroup with $\mathsf{r}(G_p) \ge 2$. Then $p = 2$, G_2 is an elementary 2-group, and $G \cong C_2^{\mathsf{r}(G)} \oplus H$, where H is a cyclic subgroup of odd order.

Let A be an atom of $\mathcal{B}(G)$ with length $|A| = \mathsf{D}(G)$. There exists an element $g_0 \in \mathrm{supp}(A)$ such that $\mathrm{ord}(g_0)$ is even and hence $g_0 = f_0 + h_0$, where $f_0 \in G_2 \setminus \{0\}$ and $h_0 \in H$. We can find $e_2, \ldots, e_{\mathsf{r}(G)}$ with $\mathrm{ord}(e_i) = 2$ for each $i \in [2, \mathsf{r}(G)]$ such that $G_2 \cong \langle f_0 \rangle \oplus \langle e_2 \rangle \oplus \ldots \oplus \langle e_{\mathsf{r}(G)} \rangle$. Then we can construct two group isomorphisms

$$\phi : G \longrightarrow G \ \text{ by } \phi(f_0) = e_2,\ \phi(e_2) = f_0,\ \phi(e_i) = e_i \text{ for each } i \in [3, \mathsf{r}(G)], \text{ and } \phi(h) = h \text{ for each } h \in H\,,$$

and

$$\psi : G \longrightarrow G \ \text{ by } \psi(f_0) = f_0 + e_2,\ \psi(e_i) = e_i \text{ for each } i \in [2, \mathsf{r}(G)], \text{ and } \psi(h) = h \text{ for each } h \in H\,.$$

It follows that $\phi(A)$ and $\psi(A)$ are atoms of length $\mathsf{D}(G)$. We consider the set

$$G_0 = \mathrm{supp}\big((-A)A\,\phi((-A)A)\,\psi((-A)A)\big)\,.$$

Obviously, we have $G_0 = -G_0$ and for every $a \in G_0$ there is some $A' \in \mathcal{A}(G_0)$ with $a \mid A'$ and $|A'| = \mathsf{D}(G)$. Thus it is sufficient to prove $\min \Delta(G_0) = 1$.

Note that $\{g_0, -g_0, \phi(g_0), \psi(g_0)\} = \{g_0, -g_0, e_2 + h_0, g_0 + e_2\} \subset G_0$. If $\mathrm{ord}(g_0) = 2$, then $h_0 = 0$ and $\mathsf{L}(g_0^2 e_2^2 (g_0+e_2)^2) = \{2, 3\}$ imply that $\min \Delta(G_0) = 1$. Suppose that $\mathrm{ord}(g_0) \ge 4$. Since

$$\mathsf{L}(g_0^{\mathrm{ord}(g_0)}(-g_0)^{\mathrm{ord}(g_0)}) = \{2, \mathrm{ord}(g_0)\} \quad \text{and} \quad \mathsf{L}(g_0^{\mathrm{ord}(g_0)-2}(g_0+e_2)^2(-g_0)^{\mathrm{ord}(g_0)}) = \{2, \mathrm{ord}(g_0)-1\}\,,$$

it follows that $\min \Delta(G_0)$ divides $\gcd\{\mathrm{ord}(g_0)-2, \mathrm{ord}(g_0)-3\} = 1$.

CASE 2: $\mathsf{r}(G) = 1$.

Let $|G| = n$ and $g \in G$ with $\mathrm{ord}(g) = n$. First, we suppose that n is odd. Then g^n and $(2g)^n$ are atoms of length $\mathsf{D}(G) = n$, and we set $G_0 = \{g, -g, 2g, -2g\}$. Then $G_0 = -G_0$ and for every $h \in G_0$ there is some $A \in \mathcal{A}(G_0)$ with $h \mid A$ and $|A| = \mathsf{D}(G)$. It is sufficient to prove that $\min \Delta(G_0) = 1$. In fact, by Lemma 3.4.1, we obtain that $\min \Delta(G_0)$ divides $\gcd\{|g^n| - 2, |g^{n-2}(2g)| - 2\} = 1$.

Now we suppose that n is even and distinguish two subcases.

CASE 2.1: $n \notin \{4, 6, 10\}$.

It is sufficient to show that $1 \in \Delta_\rho^*(G)$. We distinguish two cases.

First, suppose that there exists an odd positive divisor m of $\frac{n}{2}+1$ such that $m \ge 5$. Then $\gcd(m, n) = 1$. Let $n = m(t+1) - 2$, where $t \ge 1$. Then $A_1 = (mg)^t g^{m-2}$, $A_2 = (mg)g^{n-m}$, $A_3 = (mg)^{2t+1} g^{m-4}$, and $A_4 = g^n$ are atoms. Since $A_1^2 A_2 = A_3 A_4$, we obtain that $1 \in \Delta(\{g, -g, mg, -mg\})$. By the definition of $\Delta_\rho^*(G)$ and Lemma 3.2.1, we have that $1 \in \Delta_\rho^*(G) \subset \Delta_\rho(G)$.

Second, suppose that for every odd positive divisor m of $\frac{n}{2}+1$, we have $m \le 3$. Then $\frac{n}{2} + 1 = 2^\alpha$ or $\frac{n}{2} + 1 = 3 \cdot 2^{\alpha-1}$ where $\alpha \in \mathbb{N}$. Thus $n + 4 \in \{2(2^\alpha+1), 2(3 \cdot 2^{\alpha-1}+1)\}$. Since $n \notin \{4, 6, 10\}$, we obtain that $\alpha \ge 3$. Let $g \in G$ with $\mathrm{ord}(g) = n$, and $n + 4 = 2k$, where k is odd with $k \ge 9$. It follows that $\gcd(k, n) = 1$ and $A_5 = (kg)g^{n-k}$, $A_6 = (kg)^3 g^{2n-3k}$, $A_7 = g^n$ are atoms. Since $A_5^3 = A_6 A_7$, we have that $1 \in \Delta(\{g, -g, kg, -kg\})$.

CASE 2.2: $n \in \{4, 6, 10\}$.

We have to show that $1 \notin \Delta_\rho(G)$. If $n \in \{4,6\}$, it is easy to check $\Delta_\rho(G) = \{n-2\}$. Suppose that $n = 10$. Let

$$G_0 = \bigcup_{A \in \mathcal{A}(G) \text{ with } |A|=n} \operatorname{supp}(A) = \bigcup_{m \in [1,9] \text{ and } \gcd(m,10)=1} \{mg\} = \{g, -g, 3g, -3g\}\,.$$

Then Lemma 3.2 implies that $\min \Delta(G_0) = \min \Delta_\rho^*(G)$.

By Lemma 2.4.1, we infer that $\min \Delta_\rho^*(G) = \min \Delta_\rho(G)$. By Lemma 3.4.3, $\min \Delta(G_0) = \gcd\{\|V\|_g - 1 \mid V \in \mathcal{A}(G_0)\} = 2$ which implies that $1 \notin \Delta_\rho(G)$. □

Lemma 3.6. *Let $G = C_m \oplus C_{mn}$ with $n \ge 1$ and $m \ge 2$. A sequence S over G of length $\mathsf{D}(G) = m + mn - 1$ is a minimal zero-sum sequence if and only if it has one of the following two forms:*

- $$S = e_1^{\operatorname{ord}(e_1)-1} \prod_{i=1}^{\operatorname{ord}(e_2)} (x_i e_1 + e_2),$$
  where
  (a) *$\{e_1, e_2\}$ is a basis of G,*
  (b) *$x_1, \ldots, x_{\operatorname{ord}(e_2)} \in [0, \operatorname{ord}(e_1)-1]$ and $x_1 + \ldots + x_{\operatorname{ord}(e_2)} \equiv 1 \mod \operatorname{ord}(e_1)$.*
  In this case, we say that S is of type I(a) or I(b) according to whether $\operatorname{ord}(e_2) = m$ or $\operatorname{ord}(e_2) = mn > m$.
- $$S = f_1^{sm-1} f_2^{(n-s)m+\epsilon} \prod_{i=1}^{m-\epsilon} (-x_i f_1 + f_2),$$
  where
  (a) *$\{f_1, f_2\}$ is a generating set for G with $\operatorname{ord}(f_2) = mn$ and $\operatorname{ord}(f_1) > m$,*
  (b) *$\epsilon \in [1, m-1]$ and $s \in [1, n-1]$,*
  (c) *$x_1, \ldots, x_{m-\epsilon} \in [1, m-1]$ with $x_1 + \ldots + x_{m-\epsilon} = m-1$,*
  (d) *either $s = 1$ or $mf_1 = mf_2$, with both holding when $n = 2$, and*
  (e) *either $\epsilon \ge 2$ or $mf_1 \ne mf_2$.*
  In this case, we say that S is of type II.

Proof. The characterization of minimal zero-sum sequences of maximal length over groups of rank two was done in a series of papers by Gao, Geroldinger, Grynkiewicz, Reiher, and Schmid. For the formulation used above we refer to [16, Main Proposition 7]. □

Theorem 3.7. *Let H be a transfer Krull monoid over a finite abelian group G. If G has rank two, then $\Delta_\rho(H) = \{1\}$.*

Proof. By (3.1), it may consider $\mathcal{B}(G)$ instead of H. Let $G = C_m \oplus C_{mn}$ with $n \in \mathbb{N}$, $m \ge 2$ and S be a minimal zero-sum sequence of length $\mathsf{D}(G)$ over G. By Corollary 3.3.2, it suffices to prove that $1 \in \Delta\big(\operatorname{supp}((-S)S)\big)$. We distinguish two cases depending on Lemma 3.6.

CASE 1: $S = e_1^{\operatorname{ord}(e_1)-1} \prod_{i=1}^{\operatorname{ord}(e_2)} (x_i e_1 + e_2)$ is of type I in Lemma 3.6, where (e_1, e_2) is a basis of G.

If $x_1 = \ldots = x_{\operatorname{ord}(e_2)}$, then $\operatorname{ord}(e_2) x_1 \equiv 1 \pmod{\operatorname{ord}(e_1)}$ and hence $\gcd(\operatorname{ord}(e_1), \operatorname{ord}(e_2)) = 1$, a contradiction. Suppose that $|\{x_1, \ldots, x_{\operatorname{ord}(e_2)}\}| \ge 2$. Then there exists a subsequence $Y = y_1 \cdot \ldots \cdot y_{\operatorname{ord}(e_2)}$ of $X = x_1^2 \cdot \ldots \cdot x_{\operatorname{ord}(e_2)}^2$ such that $\sigma(Y) \not\equiv 1 \pmod{\operatorname{ord}(e_1)}$. Let $\sigma(Y) \equiv \operatorname{ord}(e_1) - a \pmod{\operatorname{ord}(e_1)}$, where $a \in [0, \operatorname{ord}(e_1) - 2]$. Then

$$T_1 = e_1^a \prod_{i=1}^{\operatorname{ord}(e_2)} (y_i e_1 + e_2) \quad \text{and} \quad T_2 = e_1^{\operatorname{ord}(e_1)-2} \prod_{i=1}^{\operatorname{ord}(e_2)} (x_i e_1 + e_2)^2 T_1^{-1}$$

are two minimal zero-sum sequences with $S^2 = e_1^{\operatorname{ord}(e_1)} \cdot T_1 \cdot T_2$ whence $1 \in \Delta\big(\operatorname{supp}((-S)S)\big)$.

CASE 2: $S = f_1^{sm-1} f_2^{(n-s)m+\epsilon} \prod_{i=1}^{m-\epsilon}(-x_i f_1 + f_2)$ is of type II in Lemma 3.6, where (f_1', f_2) is a basis with $\operatorname{ord}(f_1') = m$, $\operatorname{ord}(f_2) = mn$ and $f_1 = f_1' + \alpha f_2$, $\alpha \in [1, mn-1]$.

Since $sm - 1 + (n-s)m + \epsilon = nm + \epsilon - 1 \ge nm$, we have that $2\big((n-s)m+\epsilon\big) \ge mn$ or $2(sm-1) \ge mn$. We distinguish two subcases.

CASE 2.1: $2((n-s)m+\epsilon) \ge mn$.

Then $S^2 = f_2^{nm} \cdot f_1^{2sm-2} f_2^{nm-2sm+2\epsilon} \prod_{i=1}^{m-\epsilon}(-x_i f_1 + f_2)^2$. It suffices to prove that

$$W = f_1^{2sm-2} f_2^{nm-2sm+2\epsilon} \prod_{i=1}^{m-\epsilon}(-x_i f_1 + f_2)^2 = (f_1' + \alpha f_2)^{2sm-2} f_2^{nm-2sm+2\epsilon} \prod_{i=1}^{2m-2\epsilon}\big(-y_i f_1' + (1-\alpha y_i) f_2\big)\,,$$

where $y_1 \cdot \ldots \cdot y_{2m-2\epsilon} = x_1^2 \cdot \ldots \cdot x_{m-\epsilon}^2$, is a product of two atoms, since this implies that $1 \in \Delta\big(\operatorname{supp}((-S)S)\big)$. Note that $\sum_{i \in [1, 2m-2\epsilon]} y_i = 2(\sum_{i\in[1,m-\epsilon]}) = 2m-2$ and $|W| = mn + 2m - 2 > \mathsf D(G)$, whence W is not an atom.

Suppose that $s = 1$. Then

$$W = (f_1' + \alpha f_2)^{2m-2} \cdot \prod_{i=1}^{2m-2\epsilon}\big(-y_i f_1' + (1-\alpha y_i) f_2\big) \cdot f_2^{nm-2m+2\epsilon}\,.$$

Let T be an atom dividing W, say

$$T = (f_1' + \alpha f_2)^r \cdot \prod_{i \in I}\big(-y_i f_1' + (1-\alpha y_i) f_2\big) \cdot f_2^{r'}\,, \quad \text{where}$$

$$I \subset [1, 2m-2\epsilon],\ r \equiv \sum_{i\in I} y_i \pmod m, \quad \text{and} \quad \alpha(r - \sum_{i \in I} y_i) + |I| + r' \equiv 0 \pmod{nm}\,.$$

If $r = \sum_{i\in I} y_i$, then $|I| + r' \ge mn$ which implies that $I = [1, 2m-2\epsilon]$ and $r' = nm - 2m + 2\epsilon$. Therefore $WT^{-1} \mid (f_1' + \alpha f_2)^{2m-2-r}$, a contradiction to $\operatorname{ord}(f_1) = \operatorname{ord}(f_1' + \alpha f_2) > m$. Thus $|r - \sum_{i\in I} y_i| = m$.

Now we assume to the contrary that there exist three atoms T_1, T_2, and T_3 such that $T_1T_2T_3 \mid W$, say

$$T_1 = (f_1' + \alpha f_2)^{r_1} \cdot \prod_{i \in I_1}\big(-y_i f_1' + (1-\alpha y_i) f_2\big) \cdot f_2^{r_1'}\,,$$

$$T_2 = (f_1' + \alpha f_2)^{r_2} \cdot \prod_{i \in I_2}\big(-y_i f_1' + (1-\alpha y_i) f_2\big) \cdot f_2^{r_2'}\,,$$

$$T_3 = (f_1' + \alpha f_2)^{r_3} \cdot \prod_{i \in I_3}\big(-y_i f_1' + (1-\alpha y_i) f_2\big) \cdot f_2^{r_3'}\,.$$

Then $|r_1 - \sum_{i\in I_1} y_i| = |r_2 - \sum_{i\in I_2} y_i| = |r_3 - \sum_{i \in I_3} y_i| = m$, a contradiction to $r_1 + r_2 + r_3 \le 2m-2$ and $\sum_{i\in I_1} y_i + \sum_{i\in I_2} y_i + \sum_{i \in I_3} y_i \le 2m-2$.

Suppose that $s \ge 2$. Then $mf_1 = mf_2$ whence $\alpha m \equiv m \pmod{mn}$. Let T be an atom dividing W, say

$$T = (f_1' + \alpha f_2)^r \cdot \prod_{i \in I}\big(-y_i f_1' + (1-\alpha y_i) f_2\big) \cdot f_2^{r'}\,, \quad \text{where}$$

$$I \subset [1, 2m-2\epsilon],\ r \equiv \sum_{i\in I} y_i \pmod m, \quad \text{and} \quad \alpha(r - \sum_{i \in I} y_i) + |I| + r' \equiv 0 \pmod{nm}\,.$$

If $r = \sum_{i\in I} y_i$, then $nm \le |I| + r' \le 2m - 2\epsilon + nm - 2sm + 2\epsilon \le nm - 2sm + 2m$ which implies that $s = 1$, a contradiction.

We claim that $r - \sum_{i\in I} y_i \in \{(2s-1)m, -m\}$. If $r < \sum_{i\in I} y_i$, then $\sum_{i\in I} y_i - r = m$. We assume that $r > \sum_{i \in I} y_i$. Then $r - \sum_{i\in I} y_i \in \{m, \ldots, (2s-1)m\}$. Since $|I| + r' \le 2m - 2\epsilon + nm - 2sm + 2\epsilon = nm - 2sm + 2m$ and $\alpha m \equiv m \pmod{mn}$, we have $r - \sum_{i\in I} y_i \in \{(2s-2)m, (2s-1)m\}$. If

$r-\sum_{i\in I}y_i=(2s-2)m$, then $|I|+r'=2m-2\epsilon+nm-2sm+2\epsilon$ which infers $T=W$, a contradiction. Therefore $r-\sum_{i\in I}y_i\in\{(2s-1)m,-m\}$.

Now we assume to the contrary that there exist three atoms T_1,T_2, and T_3 such that $T_1T_2T_3\,|\,W$, say

$$T_1=(f_1'+\alpha f_2)^{r_1}\cdot\prod_{i\in I_1}(-y_if_1'+(1-\alpha y_i)f_2)\cdot f_2^{r_1'}\,,$$
$$T_2=(f_1'+\alpha f_2)^{r_2}\cdot\prod_{i\in I_2}(-y_if_1'+(1-\alpha y_i)f_2)\cdot f_2^{r_2'}\,,$$
$$T_3=(f_1'+\alpha f_2)^{r_3}\cdot\prod_{i\in I_3}(-y_if_1'+(1-\alpha y_i)f_2)\cdot f_2^{r_3'}\,.$$

Then there exist two distinct $i,j\in[1,3]$, say $i=1,j=2$, such that $r_1-\sum_{i\in I_1}y_i=r_2-\sum_{i\in I_2}y_i=(2s-1)m$. Thus $2sm-2\ge r_1+r_2\ge 2(2s-1)m$, a contradiction.

CASE 2.2: $2(sm-1)\ge mn$.

Then $2s\ge n+1$. Therefore $mf_1=mf_2$ which implies that $\alpha m\equiv m \pmod{mn}$ and $\operatorname{ord}(f_1)=mn$. Since $S^2=f_1^{nm}\cdot f_1^{2sm-nm-2}f_2^{2nm-2sm+2\epsilon}\prod_{i=1}^{m-\epsilon}(-x_if_1+f_2)^2$, it suffices to prove that

$$\begin{aligned}W&=f_1^{2sm-nm-2}f_2^{2nm-2sm+2\epsilon}\prod_{i=1}^{m-\epsilon}(-x_if_1+f_2)^2\\&=(f_1'+\alpha f_2)^{2sm-nm-2}f_2^{2nm-2sm+2\epsilon}\prod_{i=1}^{2m-2\epsilon}(-y_if_1'+(1-\alpha y_i)f_2)\,,\end{aligned}$$

where $y_1\cdot\ldots\cdot y_{2m-2\epsilon}=x_1^2\cdot\ldots\cdot x_{m-\epsilon}^2$, is a product of two atoms since this implies that $1\in\Delta\big(\operatorname{supp}((-S)S)\big)$. Note that $\sum_{i\in[1,2m-2\epsilon]}y_i=2(\sum_{i\in[1,m-\epsilon]})=2m-2$, $2sm-nm-2<mn$ and $|W|=mn+2m-2>\mathsf D(G)$ whence W is not an atom.

Let T be an atom dividing W, say

$$T=(f_1'+\alpha f_2)^r\cdot\prod_{i\in I}(-y_if_1'+(1-\alpha y_i)f_2)\cdot f_2^{r'}\,,\text{where}$$

$$I\subset[1,2m-2\epsilon],\ r\equiv\sum_{i\in I}y_i\pmod m,\qquad\text{and}\quad \alpha(r-\sum_{i\in I}y_i)+|I|+r'\equiv 0\pmod{nm}\,.$$

Suppose that $2s=n+1$. Then

$$W=(f_1'+\alpha f_2)^{m-2}f_2^{(n-1)m+2\epsilon}\prod_{i=1}^{2m-2\epsilon}(-y_if_1'+(1-\alpha y_i)f_2)\,,$$

and we assume to the contrary that there exist three atoms T_1,T_2, and T_3 such that $T_1T_2T_3\,|\,W$, say

$$T_1=(f_1'+\alpha f_2)^{r_1}\cdot\prod_{i\in I_1}(-y_if_1'+(1-\alpha y_i)f_2)\cdot f_2^{r_1'}\,,$$
$$T_2=(f_1'+\alpha f_2)^{r_2}\cdot\prod_{i\in I_2}(-y_if_1'+(1-\alpha y_i)f_2)\cdot f_2^{r_2'}\,,$$
$$T_3=(f_1'+\alpha f_2)^{r_3}\cdot\prod_{i\in I_3}(-y_if_1'+(1-\alpha y_i)f_2)\cdot f_2^{r_3'}\,.$$

Then there exist two distinct $i,j\in[1,3]$, say $i=1,j=2$, such that $r_1-\sum_{i\in I_1}y_i=r_2-\sum_{i\in I_2}y_i=0$. Thus $2nm\le|I_1|+r_1'+|I_2|+r_2'<(n-1)m+2\epsilon+2m-2\epsilon=nm+m$, a contradiction.

Suppose that $2s\ge n+2$. Consider the atom T. If $r=\sum_{i\in I}y_i$, then $nm\le|I|+r'\le 2m-2\epsilon+2nm-2sm+2\epsilon\le(2n-2s+2)m\le nm$. Therefore $I=[1,2m-2\epsilon]$ and $r'=2nm-2sm+2\epsilon$ which infers that $T=W$, a contradiction.

We claim that $r-\sum_{i\in I} y_i \in \{(2s-n-1)m, -m\}$. If $r<\sum_{i\in I} y_i$, then $\sum_{i\in I} y_i - r = m$. We assume that $r>\sum_{i\in I} y_i$. Then $r-\sum_{i\in I} y_i \in \{m,\ldots,(2s-n-1)m\}$. Since $|I|+r' \le 2m-2\epsilon+2nm-2sm+2\epsilon \le (2n-2s+2)m$ and $\alpha m \equiv m \pmod{mn}$, we have $r-\sum_{i\in I} y_i \in \{(2s-n-2)m,(2s-n-1)m\}$. If $r-\sum_{i\in I} y_i=(2s-n-2)m$, then $I=[1,2m-2\epsilon]$ and $r'=2nm-2sm+2\epsilon$ which infers that $T=W$, a contradiction. Therefore $r-\sum_{i\in I} y_i \in \{(2s-n-1)m,-m\}$.

Assume to the contrary that there exist three atoms T_1, T_2, and T_3 such that $T_1T_2T_3 \,|\, W$, say

$$T_1 = (f_1'+\alpha f_2)^{r_1} \cdot \prod_{i\in I_1}(-y_i f_1' + (1-\alpha y_i) f_2) \cdot f_2^{r_1'}\,,$$
$$T_2 = (f_1'+\alpha f_2)^{r_2} \cdot \prod_{i\in I_2}(-y_i f_1' + (1-\alpha y_i) f_2) \cdot f_2^{r_2'}\,,$$
$$T_3 = (f_1'+\alpha f_2)^{r_3} \cdot \prod_{i\in I_3}(-y_i f_1' + (1-\alpha y_i) f_2) \cdot f_2^{r_3'}\,.$$

Then there exist two distinct $i,j\in[1,3]$, say $i=1, j=2$, such that $r_1-\sum_{i\in I_1} y_i = r_2 - \sum_{i\in I_2} y_i = (2s-n-1)m$. Thus $2sm-nm-2 \ge r_1+r_2 \ge 2(2s-n-1)m$ and hence $(n+2)m-2 \ge 2sm \ge (n+2)m$, a contradiction. □

The characterization of all minimal zero-sum sequences over groups $C_2\oplus C_2\oplus C_{2n}$, as given in the next lemma, is due to Schmid ([33, Theorem 3.13]).

Lemma 3.8. *Let $G=C_2\oplus C_2\oplus C_{2n}$ with $n\ge 2$. Then $A\in\mathcal F(G)$ is a minimal zero- sum sequence of length $\mathsf D(G)$ if and only if there exists a basis (f_1,f_2,f_3) of G, where $\mathrm{ord}(f_1)=\mathrm{ord}(f_2)=2$ and $\mathrm{ord}(f_3)=2n$, such that A is equal to one of the following sequences*:

(i) $f_3^{v_3}(f_3+f_2)^{v_2}(f_3+f_1)^{v_1}(-f_3+f_2+f_1)$ *with $v_1,v_2,v_3\in\mathbb N$ odd, $v_3\ge v_2\ge v_1$, and $v_3+v_2+v_1=2n+1$.*

(ii) $f_3^{v_3}(f_3+f_2)^{v_2}(af_3+f_1)(-af_3+f_2+f_1)$ *with $v_2,v_3\in\mathbb N$ odd, $v_3\ge v_2$, $v_2+v_3=2n$, and $a\in[2,n-1]$.*

(iii) $f_3^{2n-1}(af_3+f_2)(bf_3+f_1)(cf_3+f_2+f_1)$ *with $a+b+c=2n+1$ where $a\le b\le c$ and $a,b\in[2,n-1]$, $c\in[2,2n-3]\setminus\{n,n+1\}$.*

(iv) $f_3^{2n-1-2v}(f_3+f_2)^{2v}f_2(af_3+f_1)\big((1-a)f_3+f_2+f_1\big)$ *with $v\in[0,n-1]$ and $a\in[2,n-1]$.*

(v) $f_3^{2n-2}(af_3+f_2)\big((1-a)f_3+f_2\big)(bf_3+f_1)\big((1-b)f_3+f_1\big)$ *with $a,b\in[2,n-1]$ and $a\ge b$.*

(vi) $(\prod_{i=1}^{2n}(f_3+d_i))f_2f_1$ *where $S=\prod_{i=1}^{2n}d_i\in\mathcal F(\langle f_1,f_2\rangle)$ with $\sigma(S)=f_1+f_2$.*

Theorem 3.9. *Let H be a transfer Krull monoid over a group G where $G\cong C_2\oplus C_2\oplus C_{2n}$ with $n\ge 2$. Then $\Delta_\rho(H)=\{1\}$.*

Proof. By (3.1), we may consider $\mathcal B(G)$ instead of H. Let S be a minimal zero-sum sequence of length $\mathsf D(G)$ over G. By Corollary 3.3.2, it suffices to prove that $1\in\Delta\big(\mathrm{supp}((-S)S)\big)$. We distinguish five cases induced by the structural description given by Lemma 3.8, and use Lemma 3.4.1 without further mention.

CASE 1: $S=f_3^{v_3}(f_3+f_2)^{v_2}(af_3+f_1)(-af_3+f_2+f_1)$ with $a\in[1,n-1]$ as in Lemma 3.8.(i) or (ii).

Since
$$W=f_3^{2n-1}(f_3+f_2)(af_3+f_1)(-af_3+f_2+f_1)\in\mathcal A\big(\mathrm{supp}((-S)S)\big)$$
and
$$W^2=f_3^{2n}\cdot f_3^{2n-2}(f_3+f_2)^2\cdot(af_3+f_1)^2(-af_3+f_2+f_1)^2\,,$$
we obtain that $1\in\Delta\big(\mathrm{supp}((-S)S)\big)$.

CASE 2: $S=f_3^{2n-1}(af_3+f_2)(bf_3+f_1)(cf_3+f_2+f_1)$ as in Lemma 3.8.(iii).

Suppose that $c\ge n+2$. Then $S^2=f_3^{2n}\cdot f_3^{2n-2a}(af_3+f_2)^2\cdot f_3^{2a-2}(bf_3+f_1)^2(cf_3+f_2+f_1)^2$, where $f_3^{2n-2a}(af_3+f_2)^2$ and $f_3^{2a-2}(bf_3+f_1)^2(cf_3+f_2+f_1)^2$ are atoms, and hence $1\in\Delta\big(\mathrm{supp}((-S)S)\big)$.

Suppose that $c \le n-1$. Then

$$W_1 = (-f_3)^{2a}(af_3+f_2)^2,\ W_2 = (-f_3)^{2b}(bf_3+f_1)^2,\ W_3 = (-f_3)^{2c}(cf_3+f_2+f_1)^2\,,$$

and $W = (-f_3)(af_3+f_2)(bf_3+f_1)(cf_3+f_2+f_1)$ are atoms with $W_1W_2W_3 = W^2 \cdot \big((-f_3)^{2n}\big)^2$ whence $1 \in \Delta\big(\operatorname{supp}((-S)S)\big)$.

CASE 3: $S = f_3^{2n-1-2v}(f_3+f_2)^{2v}f_2(af_3+f_1)\big((1-a)f_3+f_2+f_1\big)$ as in Lemma 3.8.(iv).

Then $\{f_3, -f_3, f_2, af_3+f_1, (1-a)f_3+f_2+f_1\} \subset \operatorname{supp}((-S)S)$. Since $W = (-f_3)f_2(af_3+f_1)\big((1-a)f_3+f_2+f_1\big)$ is an atom of length 4, we have that $\min \Delta\big(\operatorname{supp}((-S)S)\big) \mid 2$.

Setting

$$W_1 = (af_3+f_1)^2(-f_3)^{2a} \quad \text{and} \quad W_2 = \big((1-a)f_3+f_1+f_2\big)^2 f_3^{2a-2}$$

we observe that $W_1W_2(f_2)^2 = W^2(f_3(-f_3))^{2a-2}$. Therefore $\min \Delta\big(\operatorname{supp}((-S)S)\big) \mid 2a-3$ which implies that $\min \Delta\big(\operatorname{supp}((-S)S)\big) = 1$.

CASE 4: $S = f_3^{2n-2}(af_3+f_2)\big((1-a)f_3+f_2\big)(bf_3+f_1)\big((1-b)f_3+f_1\big)$ as in Lemma 3.8.(v).

Since $(-f_3)(af_3+f_2)\big((1-a)f_3+f_2\big)$ is an atom of length 3 over $\operatorname{supp}((-S)S)$, we have that $1 \in \Delta\big(\operatorname{supp}((-S)S)\big)$.

CASE 5: $S = (\prod_{i=1}^{2n}(f_3+d_i))f_2f_1$ with $T = \prod_{i=1}^{2n} d_i$ and $\sigma(T) = f_1+f_2$ as in Lemma 3.8.(vi).

Since $\sigma(T) \ne 0$, we have $|\operatorname{supp}(T)| \ge 2$, say $d_1 \ne d_2$. If $d_1+d_2 \in \{f_1, f_2\}$, then $(f_3+d_1)(-f_3+d_2)(d_1+d_2)$ is an atom of length 3 over $\operatorname{supp}((-S)S)$ which implies that $1 \in \Delta\big(\operatorname{supp}((-S)S)\big)$. If $d_1+d_2 = f_1+f_2$, then $W_1 = (f_3+d_1)(-f_3+d_2)f_1f_2$ and $W_2 = (f_3+d_1)^2(-f_3+d_2)^2$ are atoms with $W_1^2 = W \cdot f_1^2 \cdot f_2^2$ whence $1 \in \Delta\big(\operatorname{supp}((-S)S)\big)$. □

Lemma 3.10. *Let G be a finite abelian group with rank $\mathsf r(G) \ge 2$ and $\exp(G) \ge 3$, and let $U \in \mathcal A(G)$ with $|U| = \mathsf D(G)$. If there exist independent elements $e_1, \ldots, e_t$ with $t \ge 2$ and an element g such that $\{e_1, \ldots, e_t, g\} \subset \operatorname{supp}(U)$ and $ag = k_1e_1 + \ldots + k_te_t$ for some $a \in [1, \operatorname{ord}(g)-1] \setminus \{\frac{\operatorname{ord}(g)}{2}\}$ and with $k_i \in [1, \operatorname{ord}(e_i)-1]$ for all $i \in [1,t]$, then $\min \Delta\big(\operatorname{supp}((-U)U)\big) = 1$. In particular, if $\operatorname{supp}(U)$ contains a basis of G, then $\min \Delta\big(\operatorname{supp}((-U)U)\big) = 1$.*

Proof. Let $(e_1, \ldots, e_t)$ be independent with $t \ge 2$ and let $g \in G$ such that $\{e_1, \ldots, e_t, g\} \subset \operatorname{supp}(U)$ and $ag = k_1e_1 + \ldots + k_te_t$ for some $a \in [1, \operatorname{ord}(g)-1] \setminus \{\frac{\operatorname{ord}(g)}{2}\}$ and with $k_i \in [1, \operatorname{ord}(e_i)-1]$ for every $i \in [1,t]$.

Now we assume that $a \in [1, \operatorname{ord}(g)-1] \setminus \{\frac{\operatorname{ord}(g)}{2}\}$ is minimal such that $ag \in \langle e_1, \ldots, e_t\rangle$ which implies that $a \mid \operatorname{ord}(g)$ and hence $a \in [1, \lfloor \frac{\operatorname{ord}(g)}{2} \rfloor - 1]$. For every $i \in [1,t]$, we replace e_i by $-e_i$, if necessary, in order to obtain $k_i \le \operatorname{ord}(e_i)/2$. Thus we obtain that $\{e_1, \ldots, e_t\} \subset \operatorname{supp}((-U)U)$ such that $ag = k_1e_1 + \ldots + k_te_t$ with $k_i \in [1, \lfloor \operatorname{ord}(e_i)/2 \rfloor]$ for every $i \in [1,t]$. Since $a \ne \frac{\operatorname{ord}(g)}{2}$, there exists $i \in [1,t]$, say $i = 1$, such that $k_1 \ne \operatorname{ord}(e_1)/2$. Now we distinguish two cases.

CASE 1: For all $i \in [1,t]$, we have $k_i \ne \operatorname{ord}(e_i)/2$.

Then, by the minimality of a,

$$W_1 = g^a e_1^{\operatorname{ord}(e_1)-k_1} e_2^{\operatorname{ord}(e_2)-k_2} \prod_{i \in [3,t]} (-e_i)^{k_i} \quad \text{and} \quad W_2 = g^{2a} e_1^{\operatorname{ord}(e_1)-2k_1} e_2^{\operatorname{ord}(e_2)-2k_2} \prod_{i \in [3,t]} (-e_i)^{2k_i}$$

are atoms over $\operatorname{supp}((-U)U)$. Since $W_1^2 = W_2 \cdot e_1^{\operatorname{ord}(e_1)} \cdot e_2^{\operatorname{ord}(e_2)}$, we infer that $1 \in \Delta\big(\operatorname{supp}((-U)U)\big)$ which implies that $\min \Delta\big(\operatorname{supp}((-U)U)\big) = 1$.

CASE 2: There exists $i \in [2,t]$ such that $k_i = \operatorname{ord}(e_i)/2$.

After renumbering if necessary, there exists $t_0 \in [1, t-1]$ such that $k_i \ne \operatorname{ord}(e_i)/2$ for every $i \in [1, t_0]$ and $k_i = \operatorname{ord}(e_i)/2$ for every $i \in [t_0+1, t]$. Then

$$V_1 = g^a \prod_{i\in[1,t]} (-e_i)^{k_i} \quad \text{and} \quad V_2 = g^a e_1^{\operatorname{ord}(e_1)-k_1} \prod_{i\in[2,t]} (-e_i)^{k_i}$$

are atoms over $\operatorname{supp}((-U)U)$. Since

$$V_1^2 = g^{2a} \prod_{i\in[1,t_0]} (-e_i)^{2k_i} \cdot \prod_{i\in[t_0+1,t]} (-e_i)^{\operatorname{ord}(e_i)},$$

$$V_2^2 = g^{2a} e_1^{\operatorname{ord}(e_1)-2k_1} \prod_{i\in[2,t_0]} (-e_i)^{2k_i} \cdot \prod_{i\in[t_0+1,t]} (-e_i)^{\operatorname{ord}(e_i)} \cdot e_1^{\operatorname{ord}(e_1)},$$

and $g^{2a} \prod_{i\in[1,t_0]} (-e_i)^{2k_i}$, $g^{2a} e_1^{\operatorname{ord}(e_1)-2k_1} \prod_{i\in[2,t_0]} (-e_i)^{2k_i}$ are atoms, we infer that

$$\min \Delta\big(\operatorname{supp}((-U)U)\big) \mid \gcd(1+t-t_0-2, 1+t-t_0+1-1)$$

whence $\min \Delta\big(\operatorname{supp}((-U)U)\big) = 1$.

To show the in particular part, let $\{e_1, \ldots, e_t\} \subset \operatorname{supp}(U)$ be a basis of G, and note that $t \ge \mathsf r(G)$ by [17, Lemma A.6]. For each $i \in [1, t]$, we set $I_i = \{g \in \operatorname{supp}(U) \mid g \in \langle e_i \rangle\}$ and $T_i = \prod_{g\in I_i} g^{\mathsf v_g(U)}$. Then

$$U = T_1 \cdot \ldots \cdot T_t T, \quad \text{where} \quad 1 \ne T = \prod_{g\in \operatorname{supp}(U)\setminus \cup_{i\in[1,t]} I_i} g^{\mathsf v_g(U)}.$$

Therefore for every $g \in \operatorname{supp}(T)$, there exists a subset $J \subset [1, t]$ with $|J| \ge 2$ such that $g = \sum_{j\in J} k_j e_j$, where $k_j \in [1, \operatorname{ord}(e_j) - 1]$ for each $j \in J$. If $\operatorname{ord}(g) \ne 2$ for some $g \in \operatorname{supp}(T)$, then the assumptions of the main case hold whence $\min \Delta\big(\operatorname{supp}((-U)U)\big) = 1$.

Now suppose that $\operatorname{ord}(g) = 2$ for each $g \in \operatorname{supp}(T)$. Then $\sigma(T_1) \cdot \ldots \cdot \sigma(T_t)\sigma(T)$ is an atom, $\operatorname{ord}(\sigma(T)) = 2$, and $\sigma(T_i) \in \langle e_i \rangle$ for each $i \in [1, t]$. It follows that $\sigma(T_i) = \frac{\operatorname{ord}(e_i)}{2} e_i$ for each $i \in [1, t]$, $|T| = 1$, and $\sigma(T) = \frac{\operatorname{ord}(e_1)}{2} e_1 + \ldots + \frac{\operatorname{ord}(e_t)}{2} e_t$. Since $|U| = \mathsf D(G) \ge \mathsf D^*(G) \ge 1 + \sum_{j=1}^{t} (\operatorname{ord}(e_j) - 1)$ by [17, Proposition 5.1.7], we have $|T_j| = \operatorname{ord}(e_j) - 1$ for each $i \in [1, t]$. Since $\exp(G) \ge 3$, we may assume that $\operatorname{ord}(e_1) \ge 3$ after renumbering if necessary. Since $e_1 \in \operatorname{supp}(T_1)$ and T_1 is a zero-sum free sequence over $\langle e_1 \rangle$ of length $\operatorname{ord}(e_1) - 1$, we obtain $\sigma(T_1) = e_1 = \frac{\operatorname{ord}(e_1)}{2} e_1$ by [17, Theorem 5.1.10], a contradiction to $\operatorname{ord}(e_1) \ge 3$. □

Theorem 3.11. *Let H be a transfer Krull monoid over a group G where $G = C_{p^k}^r$ with $k, r \in \mathbb N$, $r \ge 2$, and $p \in \mathbb P$ such that $p^k \ge 3$. Then $\Delta_\rho(H) = \{1\}$.*

Proof. By (3.1), it is sufficient to consider $\mathcal B(G)$ instead of H. By Corollary 3.3.2, we only need to show that $\min \Delta\big(\operatorname{supp}((-U)U)\big) = 1$ for every atom $U \in \mathcal A(G)$ of length $\mathsf D(G)$. Let U be an atom of length $\mathsf D(G)$. Then $\langle \operatorname{supp}(U) \rangle = G$ by [17, Proposition 5.1.4], and hence $\operatorname{supp}(U)$ contains a basis of G by [17, Lemma A.7]. Now Lemma 3.10 implies that $\min \Delta\big(\operatorname{supp}((-U)U)\big) = 1$. □

If G is an elementary 2-group of rank $r \ge 3$, then the above assumption of Lemma 3.10 never holds true. Thus elementary 2-groups need a different approach.

Lemma 3.12. *Let G be an elementary 2-group of rank $r \ge 3$ and let $U, V \in \mathcal A(G)$ be distinct atoms of length $\mathsf D(G)$. Then $1 \in \Delta(\mathsf L(UV^2))$.*

Proof. Since U and V are distinct, there exists an element $g \in \operatorname{supp}(U) \setminus \operatorname{supp}(V)$, and clearly $\operatorname{supp}(U) \setminus \{g\}$ is a basis of G. We set $\operatorname{supp}(U) \setminus \{g\} = \{e_1, \ldots, e_r\}$, $g = e_0 = e_1 + \ldots + e_r$, and then $U = e_0 e_1 \cdot \ldots \cdot e_r$. Since $\{e_1, \ldots, e_r\}$ is a basis of G, V can be written in the form $V = e_{I_1} \cdot \ldots \cdot e_{I_{r+1}}$, where $\emptyset \ne I_j \subset [1, r]$ and $e_{I_j} = \sum_{i\in I_j} e_i$ for every $j \in [1, r+1]$. We continue with the following assertion.

A. There exist two distinct $k_1, k_2 \in [1, r+1]$ such that $I_{k_1} \cap I_{k_2} \ne \emptyset$, $I_{k_1} \setminus I_{k_2} \ne \emptyset$, and $I_{k_2} \setminus I_{k_1} \ne \emptyset$.

Proof of **A**. First, we choose I, say $I = I_1$, to be maximal in $\{I_j \mid j \in [1, r+1]\}$. Note that $e_0 \notin \mathrm{supp}(V)$ and hence $I_j \neq [1, r]$ for every $j \in [1, r+1]$. Since $I_1 \subset \cup_{j \in [2, r+1]} I_j$, we can choose $K \subset [2, r+1]$ to be minimal such that $I_1 \subset \cup_{j \in K} I_j$. Then $I \cap I_k \neq \emptyset$ and $I \setminus I_k \neq \emptyset$ for all $k \in K$. If there exists $k \in K$ such that $I_k \setminus I_1 \neq \emptyset$, then we are done. Otherwise, $I_k \subset I_1$ for all $k \in K$. By the maximality of I_1, we know that $|K| \geq 2$ and by the minimality of K, we have that $I_{k_1} \setminus I_{k_2} \neq \emptyset$ and $I_{k_2} \setminus I_{k_1} \neq \emptyset$ for every distinct k_1 and k_2. Assume to contrary that $I_{k_1} \cap I_{k_2} = \emptyset$ for every distinct k_1 and k_2. Thus $e_{I_1} \prod_{k \in K} e_{I_k}$ is an atom, a contradiction to $|V| = \mathsf{D}(G)$. □[Proof of **A**]

After renumbering if necessary, we suppose that $I_1 \cap I_2 \neq \emptyset$, $I_1 \setminus I_2 \neq \emptyset$, and $I_2 \setminus I_1 \neq \emptyset$. We define

$$W_1 = e_{I_1} e_{I_2} \prod_{i \in (I_1 \cup I_2) \setminus (I_1 \cap I_2)} e_i \,, \qquad W_2 = e_0 e_{I_1} e_{I_2} \prod_{i \notin (I_1 \cup I_2) \setminus (I_1 \cap I_2)} e_i \,,$$

and observe that W_1, W_2 are atoms. Since

$$UV^2 = U \cdot e_{I_1}^2 \cdot e_{I_2}^2 \cdot \prod_{j \in [3, r+1]} e_{I_j}^2 = W_1 \cdot W_2 \cdot \prod_{j \in [3, r+1]} e_{I_j}^2 \,,$$

we obtain that $1 \in \Delta(\mathsf{L}(UV^2))$. □

Theorem 3.13. *Let H be a transfer Krull monoid over an elementary 2-group G of rank $r \geq 2$. Then $\Delta_\rho^*(H) = \Delta_\rho(H) = \{1, r-1\}$.*

Proof. By (3.1), it is sufficient to consider $\mathcal{B}(G)$ instead of H. Let $(e_1, \ldots, e_r)$ be a basis of G and $S = e_0 e_1 \cdot \ldots \cdot e_r \in \mathcal{A}(G)$, where $e_0 = e_1 + \ldots + e_r$. Then $\Delta\big(\mathrm{supp}(S)\big) = \{r-1\}$ and hence $r-1 \in \Delta_\rho^*(G)$. By Theorem 3.5, we have that $\Delta_\rho(G) \supset \Delta_\rho^*(G) \supset \{1, r-1\}$. Thus it remains to prove that $\Delta_\rho(G) \subset \{1, r-1\}$.

Since $\max \Delta_\rho(G) \leq \max \Delta(G) = r-1$ by [17, Theorem 6.7.1], we may suppose that $r \geq 4$. Assume to the contrary that there exists $d \in \Delta_\rho(G) \setminus \{1, r-1\}$. Then for every $k \in \mathbb{N}$ there is a $B_k \in \mathcal{B}(G)$ such that $\rho\big(\mathsf{L}(B_k)\big) = \mathsf{D}(G)/2$ and $\mathsf{L}(B_k)$ is an AAP with difference d and length $\ell \geq k$. Lemma 3.2.1 implies that B_k is a product of atoms having length $\mathsf{D}(G)$. We fix $k = |\{A \in \mathcal{A}(G) \mid |A| = \mathsf{D}(G)\}| + 1$. If $B_k = U^t$ with $t \in \mathbb{N}$ for some $U \in \mathcal{A}(G)$ with $|U| = \mathsf{D}(G)$, then $r - 1 = \min \Delta\big(\mathrm{supp}(U)\big) = \min \Delta\big(\mathrm{supp}(B_k)\big) \mid d$, a contradiction. Otherwise, the choice of k implies that there are distinct atoms $U, V \in \mathcal{A}(G)$ with $|U| = |V| = \mathsf{D}(G)$ such that $U^2 V \mid B_k$. By Lemma 3.12, $1 \in \Delta(\mathsf{L}(U^2 V)) \subset \Delta(\mathsf{L}(B_k))$ and hence $d \mid 1$, a contradiction. □

Theorem 3.14. *Let H be a transfer Krull monoid over a finite cyclic group G of order $n \geq 3$. Then $n - 2 \in \Delta_\rho^*(H) = \Delta_\rho(H)$.*

Proof. By (3.1), it is sufficient to consider $\mathcal{B}(G)$ instead of H. Since $n - 2 \in \Delta_\rho^*(G) \subset \Delta_\rho(G)$, it remains to verify that $\Delta_\rho(G) \subset \Delta_\rho^*(G)$.

Let $d \in \Delta_\rho(G)$. Then for every $k \in \mathbb{N}$ there is a $B_k \in \mathcal{B}(G)$ such that $\rho\big(\mathsf{L}(B_k)\big) = \mathsf{D}(G)/2$ and $\mathsf{L}(B_k)$ is an AAP with difference d and length $\ell \geq k$. Thus $\gcd \Delta(\mathsf{L}(B_k)) = d$. We set $k = n(n-1) + 1$, $G_0 = \mathrm{supp}(B_k)$, and claim that $\min \Delta(G_0) = \gcd \Delta(\mathsf{L}(B_k))$ which implies that $d = \min \Delta(G_0) \in \Delta_\rho^*(G)$.

Clearly, $\min \Delta(G_0) \mid d$, and hence it remains to prove that $d \mid \min \Delta(G_0)$. By Lemma 3.2, B_k is a product of atoms having length $\mathsf{D}(G) = n$. Note that $|\mathrm{supp}(U)| = 1$ for all atoms of length n and $|\{U \in \mathcal{A}(G) \mid |U| = n\}| \leq n - 1$. Thus $k = n(n-1) + 1$ implies that B_k is a product of the form

$$B_k = U_1^{n+1} U_2 \cdot \ldots \cdot U_r \,,$$

where $r \in \mathbb{N}$, $U_1, \ldots, U_r$ are atoms of length n, and $U_1 = g^n$, where $g \in G$ with $\mathrm{ord}(g) = n$.

Then for every atom $V \in \mathcal{A}(G_0)$, we have $V \mid U_1 \cdot \ldots \cdot U_r$ and $\{n+1, \|V\|_g + n\} \subset \mathsf{L}(U_1^n V)$. Therefore $d \mid \|V\|_g - 1$ for all $V \in \mathcal{A}(G_0)$ whence d divides $\gcd\{\|V\|_g - 1 \mid V \in \mathcal{A}(G_0)\}$. Since $\min \Delta(G_0) = \gcd\{\|V\|_g - 1 \mid V \in \mathcal{A}(G_0)\}$ by Lemma 3.4.3, the claim follows. □

Corollary 3.15. *We have* $\Delta_\rho(C_4) = \{2\}$, $\Delta_\rho(C_5) = \{1,3\}$, $\Delta_\rho(C_6) = \{4\}$, $\Delta_\rho(C_7) = \{1,5\}$, $\Delta_\rho(C_8) = \{1,6\}$, $\Delta_\rho(C_9) = \{1,7\}$, $\Delta_\rho(C_{10}) = \{2,8\}$, $\Delta_\rho(C_{11}) = \{1,9\}$, $\Delta_\rho(C_{12}) = \{1,10\}$.

Proof. Use Theorem 3.14 and Lemma 3.4.3. □

In the next lemma we need some basics from the theory of continued fractions (see [29] for some background; in particular, we use Theorems 2.1.3 and 2.1.7 of [29]).

Lemma 3.16. *Let G be a cyclic group with order $n > 3$, $g \in G$ with* $\operatorname{ord}(g) = n$, *and* $a \in [2, n-1]$ *with* $\gcd(a,n) = 1$. *Let $[a_0, \ldots, a_m]$ be the continued fraction expansion of n/a with odd length (i.e. m is even).*

1. $\min \Delta(\{g, ag\}) = \gcd(a_1, a_3, \ldots, a_{m-1}) < n-2$ *and* $\min \Delta(\{g, -g, ag, -ag\}) \in \Delta_\rho^*(G)$.
2. *If $a < n/2$, then* $\min \Delta(\{g, ag, -ag, -g\}) = \gcd(a_0 - 1, a_1, \ldots, a_{m-1}, a_m - 1)$. *Note that this also holds for continued fraction expansion of n/a with even length and hence this holds for regular continued fraction expansion of n/a (i.e. $a_m > 1$).*

Proof. 1. For the first part, see [7, Theorem 2.1] or [14, Theorem 1]. For the second part, since g^n and $(ag)^n$ are two atoms of length $\mathsf D(G)$, we obtain $\rho\big(\mathsf L(g^n(-g)^n(ag)^n(-ag)^n)\big) = \mathsf D(G)/2$ which implies $\min \Delta(\{g, -g, ag, -ag\}) \in \Delta_\rho^*(G)$ by Lemma 3.2.3.

2. Suppose that $a < n/2$. By Lemma 3.4.3, we have

$$\begin{aligned}
&\min \Delta(\{g, ag, -ag, -g\}) \\
=&\gcd\{\|V\|_g - 1 \mid V \in \mathcal A(\{g, ag, -ag, -g\})\} \\
=&\gcd\{\|V\|_g - 1 \mid V \in \mathcal A(\{g, ag\}) \cup \mathcal A(\{g, -ag\}) \cup \mathcal A(\{-g, ag\}) \cup \mathcal A(\{-g, -ag\})\} \\
=&\gcd\{\|V\|_g - 1 \mid V \in \mathcal A(\{g, ag\}) \cup \mathcal A(\{g, -ag\})\} \\
=&\gcd\{\min \Delta(\{g, ag\}), \min \Delta(\{g, -ag\})\}\,.
\end{aligned}$$

Since the continued fraction of $\frac{n}{n-a}$ with odd length is

$$\begin{cases} [1, a_0 - 1, a_1, \ldots, a_m - 1, 1] \text{ if } a_m > 1, \\ [1, a_0 - 1, a_1, \ldots, a_{m-1} + 1] \text{ if } a_m = 1, \end{cases}$$

item 1. implies that $\min \Delta(\{g, ag\}) = \gcd(a_1, a_3, \ldots, a_{m-1})$ and

$$\min \Delta(\{g, -ag\}) = \begin{cases} \gcd(a_0 - 1, a_2, a_4, \ldots, a_m - 1) \text{ if } a_m > 1, \\ \gcd(a_0 - 1, a_2, a_4, \ldots, a_{m-2}) \text{ if } a_m = 1. \end{cases}$$

Therefore, we obtain

$$\min \Delta(\{g, ag, -ag, -g\}) = \gcd(\min \Delta(\{g, -ag\}), \min \Delta(\{g, ag\})) = \gcd(a_0 - 1, a_1, \ldots, a_{m-1}, a_m - 1)\,.$$

□

Theorem 3.17. *Let H be a transfer Krull monoid over a finite cyclic group G of order $n \ge 3$. Then the following statements are equivalent*:

(a) $\Delta_\rho^*(H) \setminus \{1, n-2\} \ne \emptyset$.

(b) *There is an* $a \in [2, \lfloor n/2 \rfloor]$ *with* $\gcd(n, a) = 1$ *such that* $\gcd(a_0 - 1, a_1, \ldots, a_{m-1}, a_m - 1) > 1$, *where* $[a_0, a_1, \ldots, a_m]$ *is the regular continued fraction expansion of n/a (i.e. $a_m > 1$).*

Proof. By (3.1), it is sufficient to prove the equivalence for $\mathcal B(G)$ instead of H.

(a) $\Rightarrow$ (b) Note that for any distinct atoms U, V of length n, we have $\min \Delta\big(\operatorname{supp}((-U)U(-V)V)\big) < n-2$ by Lemma 3.16.1. Since $\Delta_\rho^*(H) \setminus \{1, n-2\} \ne \emptyset$, there must exist distinct atoms U, V of length n such that $\min \Delta\big(\operatorname{supp}((-U)U(-V)V)\big) \in \Delta_\rho^*(G) \setminus \{1, n-2\}$. Let $U = g^n$ and $V = (ag)^n$, where $g \in G$

and $a \in [2, n-2]$ with $\gcd(n, a) = 1$. Then let $G_0 = \{g, ag, -g, -ag\}$. If $a \ge \frac{n}{2}$, then $n - a \le \frac{n}{2}$. Thus we assume that $a \le \frac{n}{2}$. Therefore Lemma 3.16.2 implies that $\gcd(a_0 - 1, a_1, \ldots, a_{m-1}, a_m - 1) > 1$, where $[a_0, a_1, \ldots, a_m]$ is the regular continued fraction expansion of n/a.

(b) $\Rightarrow$ (a) We set $G_0 = \{g, ag, -g, -ag\}$ where $g \in G$ with $\operatorname{ord}(g) = n$. Then $\min \Delta(G_0) < n - 2$ and Lemma 3.16.2 implies that $\min \Delta(G_0) > 1$. It follows that $\Delta_\rho^*(H) \setminus \{1, n-2\} \ne \emptyset$. □

Corollary 3.18. *Let G be a cyclic group of order $n > 4$, and let $g \in G$ with* $\operatorname{ord}(g) = n$.

1. *If n is even and $n - 1$ is not a prime, then there is an even $d \in \Delta_\rho^*(G) \setminus \{1, n-2\}$.*
2. *If n is even, $3 \nmid n$, and $n - 3$ is not a prime, then there is an even $d \in \Delta_\rho^*(G) \setminus \{1, n-2\}$.*
3. *If n is even and $n \equiv 2q \pmod{q^2}$ for some odd prime q with $q^2 + 2q \le n$, then there is an even $d \in \Delta_\rho^*(G) \setminus \{1, n-2\}$.*
4. *If n is even and $n \equiv q \pmod{2q+1}$ for some odd q with $5q + 2 \le n$, then there is an even $d \in \Delta_\rho^*(G) \setminus \{1, n-2\}$.*
5. *If n is even with $n \in [8, 10^9]$, then $\Delta_\rho^*(G) = \{1, n-2\}$ if and only if $n \in \{8, 12, 14, 18, 20, 30, 32, 44, 48, 54, 62, 72, 74, 84, 90, 102, 138, 182, 230, 252, 270, 450, 462, 2844\}$.*
6. *If $n > 5$ is odd and $n - 1$ is a square, then there is an odd $d \in \Delta_\rho^*(G) \setminus \{1, n-2\}$.*

Proof. Note that if $a \in [2, n-1]$ with $\gcd(a, n) = 1$, then $\min \Delta(\{g, ag, -g, -ag\}) \in \Delta_\rho^*(G)$ and $\min \Delta(\{g, ag, -g, -ag\}) < n - 2$ by Lemma 3.16.1.

1. Let $n = mt + 1$ be even with $m \in [2, n-2]$, and set $G_0 = \{g, mg, -mg, -g\}$, . Then m, t are odd, $\gcd(m, n) = 1$, and $m < n/2$. Since $[t, m]$ is the regular continued fraction of n/m, we have that $\min \Delta(G_0) = \gcd(m - 1, t - 1)$ is even and hence $\min \Delta(G_0) \in \Delta_\rho^*(G) \setminus \{1, n-2\}$.

2. If $n \equiv 1 \pmod 3$, then $n - 1$ is not a prime and hence 1. implies the assertion. Suppose $n \equiv 2 \pmod 3$ and let $n - 3 = m_1 m_2$ with $1 < m_1 < n - 3$. Then there exists $i \in [1, 2]$, say $i = 1$, such that $m_1 \equiv 1 \pmod 3$. Set $G_0 = \{g, m_1 g, -m_1 g, -g\}$. Since n is even, we obtain that m_1, m_2 are odd and hence $\lfloor \frac{m_1}{3} \rfloor$ is even. Since $[m_2, \lfloor \frac{m_1}{3} \rfloor, 3]$ is the regular continued fraction of n/m, we have that $\min \Delta(G_0) = \gcd(m_2 - 1, \lfloor \frac{m_1}{3} \rfloor, 2) = 2$ by Lemma 3.16.1 and hence $\min \Delta(G_0) \in \Delta_\rho^*(G) \setminus \{1, n-2\}$.

3. Let $n = q^2 t + 2q$ be even with $m = qt + 1$, and set $G_0 = \{g, mg, -mg, -g\}$. Then $n = qm + q$ and $t \ge 1$ is even. Since $[q, t, q]$ is the regular continued fraction of n/m, we have that $\min \Delta(G_0) = \gcd(q-1, t, q-1)$ is even by Lemma 3.16.1 and hence $\min \Delta(G_0) \in \Delta_\rho^*(G) \setminus \{1, n-2\}$.

4. Let $n = (2q + 1)t + q$ be even with t odd, and set $G_0 = \{g, (2q+1)g, -(2q+1)g, -g\}$. Then $\gcd(2q + 1, n) = 1$ and $5q + 2 \le n$ implies that $2q + 1 < n/2$. Since $[t, 2, q]$ is the regular continued fraction of $n/(2q + 1)$, we have that $\min \Delta(G_0) = \gcd(t - 1, 2, q - 1) = 2$ by Lemma 3.16.1 and hence $\min \Delta(G_0) \in \Delta_\rho^*(G) \setminus \{1, n-2\}$.

5. This was done by a computer program.

6. Let $n = m^2 + 1$ be odd, and set $G_0 = \{g, mg, -mg, -g\}$. Then m is even. Since $[m, m]$ is the regular continued fraction of n/m, we have that $\min \Delta(G_0) = \gcd(m - 1, m - 1) = m - 1 > 1$ is odd by Lemma 3.16.1 and hence $\min \Delta(G_0) \in \Delta_\rho^*(G) \setminus \{1, n-2\}$. □

Next we discuss an application of Theorem 3.17 to the so-called Characterization Problem which is in the center of all arithmetical investigations of transfer Krull monoids. It asks whether two finite abelian groups G with $\mathsf{D}(G) \ge 4$ and G', whose systems of sets of lengths $\mathcal{L}(G)$ and $\mathcal{L}(G')$ coincide, have to be isomorphic (for an overview on this topic we refer [15, Section 6]). It is well-known that for every $n \ge 4$, the systems $\mathcal{L}(C_n)$ and $\mathcal{L}(C_2^{n-1})$ are distinct and that $\mathcal{L}(C_2^{n-1}) \not\subset \mathcal{L}(C_n)$ ([21, Theorem 3.5]). If $n \in [4, 5]$, then $\mathcal{L}(C_n) \subset \mathcal{L}(C_2^{n-1})$ ([21, Section 4]), but for $n \ge 6$ there is no information available so far. The results of the present section yield the following corollary.

Corollary 3.19. *Let G be a cyclic group of order $n \ge 6$. If the equivalent statements in Theorem 3.17 hold, then $\mathcal{L}(C_n) \not\subset \mathcal{L}(C_2^{n-1})$.*

Comment. Note that Corollary 3.18 shows that the equivalent statements in Theorem 3.17 hold true for infinitely many $n \in \mathbb{N}$.

Proof. Assume to the contrary that $\mathcal{L}(C_n) \subset \mathcal{L}(C_2^{n-1})$. Then $\Delta_\rho(C_n) \subset \Delta_\rho(C_2^{n-1})$. Since $\Delta_\rho(C_2^{n-1}) = \{1, n-2\}$ by Theorem 3.13, we obtain a contradiction to Theorem 3.17. □

We end this section with the following conjecture (note, if G is cyclic of order three or isomorphic to $C_2 \oplus C_2$, then $\Delta_\rho(G) = \{1\}$).

Conjecture 3.20. *Let H be a transfer Krull monoid over a finite abelian group G with $|G| > 4$. Then $\Delta_\rho(H) = \{1\}$ if and only if G is neither cyclic nor an elementary 2-group.*

We summarize what follows so far by the results of the present section. Clearly, one implication of Conjecture 3.20 holds true. Indeed, if G is cyclic or an elementary 2-group with $|G| > 4$, then $\Delta_\rho(H) \neq \{1\}$ by Theorems 3.13 and 3.14. Conversely, for groups of rank two, and for groups isomorphic either to $C_2 \oplus C_2 \oplus C_{2n}$ or to $C_{p^k}^r$, where $n, r \geq 2$, $k \geq 1$, and p is a prime with $p^k \geq 3$, the conjecture holds true by Theorems 3.7, 3.9, and 3.11 (consequently, the conjecture holds true for all groups G with $|G| \in [5, 47]$). In view of our discussion (preceding Lemma 3.2) on the state of the art on the Davenport constant, Conjecture 3.20 might seem to be quite bold, but it is consistent with all what we know on the Davenport constant so far. Indeed, let $U \in \mathcal{A}(G)$ with $|U| = \mathsf{D}(G)$. The goal is to show that $\min \Delta\big(\operatorname{supp}((-U)U)\big) = 1$. By [17, Proposition 5.1.11], $\operatorname{supp}(U)$ contains a generating set of G. If it contains a basis, then we are done by Lemma 3.10. Suppose G is as in (3.2) with $\mathsf{D}(G) = \mathsf{D}^*(G)$, $\mathsf{r}(G) = r > 1$, and $(e_1, \ldots, e_r)$ is a basis with $\operatorname{ord}(e_i) = n_i$ for all $i \in [1, r]$. Then

$$U = e_1^{n_1-1} \cdot \ldots \cdot e_r^{n_r-1}(e_1 + \ldots + e_r)$$

is the canonical example of a minimal zero-sum sequence of length $\mathsf{D}^*(G)$. Clearly, there are minimal zero-sum sequences of different form (as Lemma 3.6 shows for $r = 2$) but their support can only be greater than or equal to $\mathsf{r}(G) + 1$ (recall that $\mathsf{r}(G) = \min\{|G_0| \mid G_0 \subset G \text{ is a generating set}\}$ by [17, Lemma A.6]). Furthermore, for subsets $G_0 \subset G_1$ of G, we have $\min \Delta(G_1) \leq \min \Delta(G_0)$. The combination of these two facts provides strong support for the above conjecture.

4. Weakly Krull monoids

The main goal in this section is to study the set $\Delta_\rho(\cdot)$ for v-noetherian weakly Krull monoids and for their monoids of v-invertible v-ideals. Our main result is given by Theorem 4.4.

We start with the local case, namely with finitely primary monoids. A monoid H is said to be *finitely primary* if there are $s, \alpha \in \mathbb{N}$ and a factorial monoid $F = F^\times \times \mathcal{F}(\{p_1, \ldots, p_s\})$ such that $H \subset F$ with

$$H \setminus H^\times \subset p_1 \cdot \ldots \cdot p_s F \quad \text{and} \quad (p_1 \cdot \ldots \cdot p_s)^\alpha F \subset H \,. \tag{4.1}$$

In this case s is called the rank of H and α is called an exponent of H. It is well-known ([17, Theorems 2.9.2 and 3.1.5]) that F is the complete integral closure of H, that

$$H \text{ has finite elasticity if and only if } s = 1 \,, \tag{4.2}$$

and that

$$H/H^\times \text{ is finitely generated if and only if } s = 1 \text{ and } (F^\times \colon H^\times) < \infty \,. \tag{4.3}$$

To provide some examples of finitely primary monoids, we first recall that every numerical monoid $H \subsetneq (\mathbb{N}_0, +)$ is finitely generated and finitely primary of rank one with accepted elasticity $\rho(H) > 1$. Furthermore, if R is a one-dimensional local Mori domain, $\widehat{R}$ its complete integral closure, and $(R \colon \widehat{R}) \neq \{0\}$, then its multiplicative monoid of non-zero elements is finitely primary ([17, Sections 2.9, 2.10, and 3.1]). Note that a finitely primary monoid H with $\rho(H) > 1$ is not a transfer Krull monoid by [21, Theorem 5.5].

Our first lemma is known for numerical monoids ([9, Theorem 2.1] and [6, Proposition 2.9]).

Lemma 4.1. *Let $H \subset F = F^\times \times \mathcal{F}(\{p\})$ be a finitely primary monoid of rank 1 and exponent α, and let $\mathsf{v} = \mathsf{v}_p \colon H \to \mathbb{N}_0$ denote the homomorphism onto the value semigroup of H. Suppose that $\{\mathsf{v}(a) \mid a \in \mathcal{A}(H)\} = \{n_1, \ldots, n_s\}$ with $1 \le n_1 < \ldots < n_s$. Then $\mathsf{v}(H) \subset \mathbb{N}_0$ is a numerical monoid, and we have*

1. *$\rho(H) = n_s/n_1$, and if $F^\times/H^\times$ is a torsion group, then the elasticity is accepted.*
2. *Let $d = \gcd\{n_i - n_{i-1} \mid i \in [2, s]\}$. Then $d \mid \gcd \Delta(H)$ and if $|F^\times/H^\times| = 1$, then $d = \gcd \Delta(H)$.*

Proof. If $a \in \mathcal{A}(H)$, then $p^\alpha F \subset H$ (see (4.1)) implies $\mathsf{v}(a) \le 2\alpha - 1$, and hence $n_s \le 2\alpha - 1$. Since $\mathbb{N}_{\ge \alpha} \subset \mathsf{v}(H)$, it follows that $\mathsf{v}(H) \subset \mathbb{N}_0$ is a numerical monoid.

1. To show that $\rho(H) \le n_s/n_1$, let $a \in H$ be given and suppose that $a = u_1 \cdot \ldots \cdot u_k = v_1 \cdot \ldots \cdot v_\ell$ where $k, \ell \in \mathbb{N}$ and $u_1, \ldots, u_k, v_1, \ldots, v_\ell \in \mathcal{A}(H)$. Then

$$\ell n_1 \le \sum_{i=1}^{\ell} \mathsf{v}(v_i) = \mathsf{v}(a) = \sum_{i=1}^{k} \mathsf{v}(u_i) \le k n_s \,,$$

whence $\ell/k \le n_s/n_1$ and thus $\rho(\mathsf{L}(a)) \le n_s/n_1$.

To show that $\rho(H) = n_s/n_1$, let $u_1 = \epsilon_1 p^{n_1}, u_2 = \epsilon_2 p^{n_s} \in \mathcal{A}(H)$ with $\epsilon_1, \epsilon_2 \in F^\times$, and let $s \in \mathbb{N}_0$ such that $s n_1 n_s \ge \alpha$. Then for every $k > s$ we have

$$u_2^{kn_1} = \epsilon_2^{kn_1} p^{kn_1 n_s} = \bigl(\epsilon_2^{kn_1} \epsilon_1^{-(k-s)n_s} p^{s n_1 n_s}\bigr)\bigl(\epsilon_1 p^{n_1}\bigr)^{(k-s)n_s} = \bigl(\epsilon_2^{kn_1} \epsilon_1^{-(k-s)n_s} p^{s n_1 n_s}\bigr) u_1^{(k-s)n_s} \,.$$

Thus

$$\rho(\mathsf{L}(u_2^{kn_1})) = \frac{\max \mathsf{L}(u_2^{kn_1})}{\min \mathsf{L}(u_2^{kn_1})} \ge \frac{1 + (k-s)n_s}{kn_1}$$

tends to n_s/n_1 as k tends to infinity.

Now suppose that $F^\times/H^\times$ is a torsion group, and let u_1, u_2 be as above. Then there is a $k_0 \in \mathbb{N}$ such that $(\epsilon_2^{n_1} \epsilon_1^{-n_s})^{k_0} \in H^\times$. Then the above calculation with $k = k_0$ and $s = 0$ shows that $\rho(\mathsf{L}(u_2^{k_0 n_1})) = n_s/n_1$.

2. For every $i \in [1, s]$ there are $t_i \in \mathbb{N}_0$ such that $n_i = n_1 + t_i d$. Since $p^\alpha F \subset H$, it follows that $\gcd(n_1, d) = 1$. Let $a \in H$ and consider two factorizations

$$a = \prod_{i=1}^{s} \prod_{j=1}^{k_i} u_{i,j} = \prod_{i=1}^{s} \prod_{j=1}^{\ell_i} v_{i,j} \,,$$

where all $u_{i,j}, v_{i,j}$ are (not necessarily distinct) atoms with $\mathsf{v}(u_{i,j}) = n_i = \mathsf{v}(v_{i,j})$ for all $i \in [1, s]$. Then

$$\mathsf{v}(a) = \sum_{i=1}^{s} k_i n_i = \sum_{i=1}^{s} \ell_i n_i = \sum_{i=1}^{s} \ell_i (n_1 + t_i d)$$

whence

$$n_1 \sum_{i=1}^{s} (\ell_i - k_i) = d \sum_{i=1}^{s} (k_i - \ell_i) t_i$$

and this implies that d divides $\sum_{i=1}^{s} (\ell_i - k_i)$. Thus d divides $\gcd \Delta(H) = \min \Delta(H)$.

Now suppose that $F^\times = H^\times$. We show that $\gcd \Delta(H)$ divides $n_i - n_{i-1}$ for every $i \in [2, s]$ which implies that $\gcd \Delta(H)$ divides d and equality follows. Let $i \in [2, s]$. Then there are atoms $u_{i-1} = \epsilon_{i-1} p^{n_{i-1}}$ and $u_i = \epsilon_i p^{n_i}$ with $\epsilon_{i-1}, \epsilon_i \in F^\times = H^\times$. Then

$$u_i^{n_{i-1}} = \bigl(\epsilon_i p^{n_i}\bigr)^{n_{i-1}} = \bigl(\epsilon_{i-1} p^{n_{i-1}}\bigr)^{n_i} (\epsilon_i^{n_{i-1}} \epsilon_{i-1}^{-n_i}) = u_{i-1}^{n_i} \eta \,,$$

where $\eta = \epsilon_i^{n_{i-1}} \epsilon_{i-1}^{-n_i} \in H^\times$. Thus $\gcd \Delta(H)$ divides $n_i - n_{i-1}$. □

We continue with simple examples showing that the elasticity need not be accepted if $F^\times/H^\times$ fails to be a torsion group, and that d need not be equal to $\min \Delta(H)$.

Example 4.2.
1. Let $H \subset F$ be a finitely primary monoid as in (4.1), and generated by $\{\epsilon_1 p^2, \epsilon_2 p^4, \epsilon p^3 \mid \epsilon \in F^\times\}$, where $\epsilon_1, \epsilon_2 \in F^\times$ with $\operatorname{ord}(\epsilon_1) = \infty$ and $\operatorname{ord}(\epsilon_2) < \infty$. We assert that $\rho(H)$ is not accepted.

First, we observe that $\mathcal{A}(H) = \{\epsilon_1 p^2, \epsilon_2 p^4, \epsilon p^3 \mid \epsilon \in F^\times\}$. Thus Lemma 4.1.1 implies that $\rho(H) = 2$. For every $b \in H$, we have $\mathsf{v}(b) \le 4 \min \mathsf{L}(b)$ and $\mathsf{v}(b) \ge 2 \max \mathsf{L}(b)$ which infer that $\rho(\mathsf{L}(b)) \le 2$. Assume to the contrary that $\rho(\mathsf{L}(b)) = 2$. Then $\mathsf{v}(b) = 4 \min \mathsf{L}(b) = 2 \max \mathsf{L}(b)$ which implies that $b = (\epsilon_2 p^4)^{\min \mathsf{L}(b)} = (\epsilon_1 p^2)^{\max \mathsf{L}(b)}$. It follows that $\epsilon_2^{\min \mathsf{L}(b)} = \epsilon_1^{2 \min \mathsf{L}(b)}$, a contradiction to our assumption on $\operatorname{ord}(\epsilon_1)$ and $\operatorname{ord}(\epsilon_2)$. Therefore $\rho(\mathsf{L}(b)) < 2$ for all $b \in H$ whence $\rho(H)$ is not accepted.

2. Let $F^\times = \{\epsilon\}$ with $\epsilon^2 = 1$, and $H = \langle \epsilon p^3, p^5 \rangle \subset F = F^\times \times \mathcal{F}(\{p\})$. Then $\min \Delta(H) = 4 > 2 = d$, where d as in Lemma 4.1.2.

Lemma 4.3.
1. *Let H be a finitely primary monoid with accepted elasticity $\rho(H) > 1$. Then $\Delta_\rho^*(H) = \Delta_\rho(H) = \Delta_1(H) = \{\min \Delta(H)\}$.*
2. *Let $H = H_1 \times \ldots \times H_n$ where $n \in \mathbb{N}$ and H_i is a finitely primary monoid with accepted elasticity and $\min \Delta(H_i) = d_i$ for all $i \in [1,n]$. Suppose that $\rho(H_1) = \ldots = \rho(H_s) = \rho(H) > \rho(H_i)$ for all $i \in [s+1,n]$. Then $\min \Delta_\rho(H) = \min \Delta_\rho^*(H) = \gcd(d_1, \ldots, d_s)$, $\max \Delta_\rho(H) = \max \Delta_\rho^*(H)$, and*

$$\Big\{ \gcd\{d_i \mid i \in I\} \ \mid \emptyset \ne I \subset [1,s] \Big\} = \Delta_\rho^*(H) \ \subset \ \Delta_\rho(H) \subset \Big\{ d \in \mathbb{N} \mid d \text{ divides some } d' \in \Delta_\rho^*(H) \Big\} .$$

Proof. 1. By Lemmas 2.2 and 2.4, we have

$$\{\min \Delta(\llbracket a \rrbracket) \mid a \in H \text{ with } \rho(\mathsf{L}(a)) = \rho(H)\} = \Delta_\rho^*(H) \subset \Delta_\rho(H) \subset \Delta_1(H) .$$

If $a \in H$ with $\rho(\mathsf{L}(a)) = \rho(H) > 1$, then $a \in H \setminus H^\times$ and hence $\llbracket a \rrbracket = H$. Thus it remains to show that $\Delta_1(H) = \{\min \Delta(H)\}$, which follows from [17, Theorem 4.3.6].

2. Without restriction we may suppose that H is reduced. Then also $H_1, \ldots, H_n$ are reduced. We use Lemma 2.6. Note that $H_1, \ldots, H_n$ need not be finitely generated whence Lemma 2.4.3 cannot be applied to the present setting.

Let $a = a_1 \cdot \ldots \cdot a_n \in H$ with $a_i \in H_i$ for all $i \in [1,n]$. If $\rho(\mathsf{L}(a)) = \rho(H)$, then $a_{s+1} = \ldots = a_n = 1$ and

$$\llbracket a \rrbracket = \prod_{i \in [1,s], a_i \ne 1} H_i .$$

For every $i \in [1,s]$, 1. implies that $\Delta_\rho(H_i) = \{d_i\}$. If $\emptyset \ne I \subset [1,s]$, then [17, Proposition 1.4.5] implies that

$$\gcd \Delta\Big(\prod_{i \in I} H_i\Big) = \gcd \bigcup_{i \in I} \Delta(H_i) ,$$

and clearly

$$\gcd \bigcup_{i \in I} \Delta(H_i) = \gcd \{\gcd \Delta(H_i) \mid i \in I\} = \gcd\{d_i \mid i \in I\} .$$

Thus we obtain that (the first equality follows from Lemma 2.2.2)

$$\begin{aligned} \Delta_\rho^*(H) &= \Big\{ \gcd \Delta(\llbracket a \rrbracket) \mid a \in H \text{ with } \rho(\mathsf{L}(a)) = \rho(H) \Big\} \\ &= \Big\{ \gcd \Delta(\prod_{i \in I} H_i) \mid \emptyset \ne I \subset [1,s] \Big\} \\ &= \Big\{ \gcd\{d_i \mid i \in I\} \mid \emptyset \ne I \subset [1,s] \Big\} . \end{aligned}$$

Since $\Delta_\rho(H) = \Delta_\rho(H_1 \times \ldots \times H_s)$, $\min \Delta(H_1 \times \ldots \times H_s) = \gcd(d_1, \ldots, d_s)$, and $\min \Delta_\rho^*(H) = \gcd(d_1, \ldots, d_s)$, it follows that $\min \Delta_\rho(H) = \gcd(d_1, \ldots, d_s)$.

Lemma 2.4.1 implies that $\Delta_\rho^*(H) \subset \Delta_\rho(H)$, and it remains to show that $\Delta_\rho(H) \subset \big\{ d \in \mathbb{N} \mid d$ divides some $d' \in \Delta_\rho^*(H) \big\}$. If this holds, then we immediately get that $\max \Delta_\rho(H) = \max \Delta_\rho^*(H)$. Now let $d \in \Delta_\rho(H)$ be given. We claim that d divides some element from $\Delta_\rho^*(H)$.

For every $k \in \mathbb{N}$ there is some $a^{(k)} \in H$ such that $\mathsf{L}(a^{(k)})$ is an AAP with difference d, length at least k, and with $\rho(\mathsf{L}(a^{(k)}) = \rho(H)$. Let $k \in \mathbb{N}$. Then $a^{(k)} = a_1^{(k)} \cdot \ldots \cdot a_s^{(k)}$ with $a_i^{(k)} \in H_i$ and $\rho\big(\mathsf{L}(a_i^{(k)})\big) = \rho(H_i) = \rho(H)$ for all $i \in [1, s]$. Then there is a subsequence $b^{(\ell)} = a^{(k_\ell)}$ of $a^{(k)}$, a nonempty subset $I \subset [1, s]$, say $I = [1, r]$, and a constant M such that the following holds for every $k \in \mathbb{N}$.

- For every $i \in [1, r]$, $\mathsf{L}(b_i^{(k)})$ is an AAP with difference d_i, length at least k, and with $\rho(\mathsf{L}(B_i^{(k)}) = \rho(H)$.
- For every $i \in [r+1, s]$, $|\mathsf{L}(b_i^{(k)})| \le M$.

Thus $\mathsf{L}(b_1^{(k)} \cdot \ldots \cdot b_r^{(k)}) = \mathsf{L}(b_1^{(k)}) + \ldots + \mathsf{L}(b_r^{(k)})$ is an AAP with difference $\gcd(d_1, \ldots, d_r) \in \Delta_\rho^*(H)$ and length growing with k. Since $\mathsf{L}(b^{(k)})$ is an AAP with difference d, it follows that d divides $\gcd(d_1, \ldots, d_r)$. $\square$

For our discussion of weakly Krull monoids we put together some notation and gather their main properties. For any undefined notion we refer to [28, 17]. In the remainder of this sections all monoids are commutative and cancellative and by a domain we always mean a commutative integral domain. If R is a domain, then its semigroup $R^\bullet = R \setminus \{0\}$ of non-zero elements is a monoid.

Let H be a monoid. Then $\mathsf{q}(H)$ denotes its quotient group,

$$\widehat{H} = \{ x \in \mathsf{q}(H) \mid \text{there is a } c \in H \text{ such that } cx^n \in H \text{ for all } n \in \mathbb{N} \} \subset \mathsf{q}(H),$$

its complete integral closure, and $(H \colon \widehat{H}) = \{ x \in \mathsf{q}(H) \mid x\widehat{H} \subset H \}$ the conductor of H. Furthermore, $H_{\text{red}} = \{ aH^\times \mid a \in H \}$ is the associated reduced monoid of H and $\mathfrak{X}(H)$ is the set of minimal non-empty prime s-ideals of H. Let $\mathcal{I}_v^*(H)$ denote the monoid of v-invertible v-ideals of H (together with v-multiplication). Then $\mathcal{F}_v(H)^\times = \mathsf{q}\big(\mathcal{I}_v^*(H)\big)$ is the quotient group of fractional v-invertible v-ideals, and $\mathcal{C}_v(H) = \mathcal{F}_v(H)^\times / \{ xH \mid x \in \mathsf{q}(H) \}$ is the v-class group of H.

The monoid H is said to be *weakly Krull* ([28, Corollary 22.5]) if

$$H = \bigcap_{\mathfrak{p} \in \mathfrak{X}(H)} H_\mathfrak{p} \quad \text{and} \quad \{ \mathfrak{p} \in \mathfrak{X}(H) \mid a \in \mathfrak{p} \} \text{ is finite for all } a \in H.$$

If H is v-noetherian, then H is weakly Krull if and only if v-$\max(H) = \mathfrak{X}(H)$ ([28, Theorem 24.5]). A domain R is *weakly Krull* if $R^\bullet$ is a weakly Krull monoid. Weakly Krull domains were introduced by Anderson, Anderson, Mott, and Zafrullah ([1, 2]), and weakly Krull monoids by Halter-Koch ([26]). The monoid H is Krull if and only if H is weakly Krull and $H_\mathfrak{p}$ is a discrete valuation monoid for each $\mathfrak{p} \in \mathfrak{X}(H)$.

Every saturated submonoid H of a monoid $D = \mathcal{F}(P) \times D_1 \ldots \times D_n$, where P is a set of primes and $D_1, \ldots, D_n$ are primary monoids, is weakly Krull if the class group $\mathsf{q}(D) / \big(D^\times \mathsf{q}(H)\big)$ is a torsion group ([19, Lemma 5.2]). We mention a few key examples examples of v-noetherian weakly Krull monoids and domains and refer to [19, Examples 5.7] for a detailed discussion. Suppose that H is as in Theorem 4.4. Then, by the previous remark, its monoid of v-invertible v-ideals $\mathcal{I}_v^*(H)$ is a weakly Krull monoid. Furthermore, all one-dimensional noetherian domains are v-noetherian weakly Krull. If R is v-noetherian weakly Krull domain with non-zero conductor $(R \colon \widehat{R})$ and $\mathfrak{p} \in \mathfrak{X}(R)$, then $R_\mathfrak{p}^\bullet$ is finitely primary, and thus the assumption made in Theorem 4.4 holds. Orders in algebraic number fields are one-dimensional noetherian and hence they are v-noetherian weakly Krull domains. If R is an order, then its v-class group $\mathcal{C}_v(R)$ (which coincides with the Picard group) as well as the index of the unit groups $(\widehat{R}^\times \colon R^\times)$ are finite and every class contains a minimal prime ideal $\mathfrak{p} \in \mathcal{P}$. Thus all assumptions made in Theorem 4.4.4 are satisfied. It was first proved by Halter-Koch ([27, Corollary 4]) that the elasticity of orders in number fields is accepted whenever it is finite.

Theorem 4.4. *Let H be a v-noetherian weakly Krull monoid with conductor $\emptyset \ne \mathfrak f = (H : \widehat H) \subsetneq H$ such that $H_{\mathfrak p}$ is finitely primary for each $\mathfrak p \in \mathfrak X(H)$. Let $\mathcal P^* = \{\mathfrak p \in \mathfrak X(H) \mid \mathfrak p \supset \mathfrak f\}$, $\mathcal P = \mathfrak X(H) \setminus \mathcal P^*$, and let $\pi \colon \mathfrak X(\widehat H) \to \mathfrak X(H)$ be the natural map defined by $\pi(\mathfrak P) = \mathfrak P \cap H$ for all $\mathfrak P \in \mathfrak X(\widehat H)$.*

1. *$\mathcal I_v^*(H)$ has finite elasticity if and only if π is bijective.*
2. *If π is bijective and $\widehat H_{\mathfrak p}^\times / H_{\mathfrak p}^\times$ are torsion groups for all $\mathfrak p \in \mathcal P^*$, then $\mathcal I_v^*(H)$ has accepted elasticity.*
3. *Suppose that $\mathcal I_v^*(H)$ has accepted elasticity, and let $\mathfrak p_1, \ldots, \mathfrak p_s \in \mathcal P^*$ be the minimal prime ideals with $\rho\big(H_{\mathfrak p_i}\big) = \rho\big(\mathcal I_v^*(H)\big)$ for all $i \in [1, s]$, and set $d_i = \min \Delta(H_{\mathfrak p_i})$. Then*
$$\big\{ \gcd\{d_i \mid i \in I\} \ \mid \emptyset \ne I \subset [1,s]\big\} = \Delta_\rho^*\big(\mathcal I_v^*(H)\big) \ \subset \ \Delta_\rho\big(\mathcal I_v^*(H)\big) \subset \big\{ d \in \mathbb N \mid d \text{ divides some } d' \in \Delta_\rho^*\big(\mathcal I_v^*(H)\big)\big\} \,.$$
4. *Let $G_{\mathcal P} \subset \mathcal C_v(H)$ denote the set of classes containing a minimal prime ideal from $\mathcal P$. Suppose that π is bijective, and that $\mathcal C_v(H)$ and $\widehat H^\times / H^\times$ are both finite. Then H has accepted elasticity and if $\rho(H) = \rho(G_{\mathcal P})$, then $\Delta_\rho(G_{\mathcal P}) \subset \Delta_\rho(H)$.*

Proof. By [19, Section 5]), we infer that $\widehat H$ is Krull, $\mathcal P^*$ is finite, and that

$$\mathcal I_v^*(H) \overset{\sim}{\rightarrow} \mathcal F(\mathcal P) \times T \,, \quad \text{where} \quad T = \prod_{\mathfrak p \in \mathcal P^*} (H_{\mathfrak p})_{\rm red} \,. \tag{4.4}$$

1. This follows from (4.2), from (4.4), and from Lemma 2.6.1.
2. This follows from Lemma 2.6.1 and from Lemma 4.1.1.
3. This follows from (4.4) and from Lemma 4.3.2.
4. There is a transfer homomorphism

$$\boldsymbol\beta \colon H \to \mathcal B(H) \,, \quad \text{where} \quad \mathcal B(H) \hookrightarrow \mathcal F(G_{\mathcal P}) \times T$$

is the T-block monoid of H and the inclusion is saturated and cofinal ([17, Definition 3.4.9]). Thus $\mathcal L\big(\mathcal B(H)\big) = \mathcal L(H)$, whence it suffices to prove all the statements for $\mathcal B(H)$ instead of proving them for H.

Since $\mathcal C_v(H)$ and $\widehat H^\times / H^\times$ are finite, the exact sequence ([19, Proposition 5.4])

$$1 \to \widehat H^\times / H^\times \to \coprod_{\mathfrak p \in \mathfrak X(H)} \widehat H_{\mathfrak p}^\times / H_{\mathfrak p}^\times \to \mathcal C_v(H) \to \mathcal C_v(\widehat H) \to 0 \,,$$

implies that $(\widehat H_{\mathfrak p}^\times : H_{\mathfrak p}^\times) < \infty$ for all $\mathfrak p \in \mathcal P^*$. Thus, by 4.3, all factors of T are finitely generated and hence T is finitely generated. Therefore $\mathcal B(H)$ is finitely generated (as a saturated submonoid of a finitely generated monoid) and hence $\mathcal B(H)$ has accepted elasticity by [17, Theorem 3.1.4].

Since $\mathcal B(G_{\mathcal P}) \subset \mathcal B(H)$ is a divisor-closed submonoid, the remaining statement follows from Lemma 2.4.2. □

Remarks 4.5.

1. Let H be as in Theorem 4.4. If π is bijective and H is seminormal, then $\mathcal I_v^*(H)$ is half-factorial ([19, Theorem 5.8.1.(a)]) and hence $\Delta\big(\mathcal I_v^*(H)\big) = \emptyset$.

2. Let R be a noetherian weakly Krull domain such that its integral closure $\overline R$ is a finitely generated R-module. Then, for $\mathfrak p \in \mathcal P^*$, the index $(\overline R_{\mathfrak p}^\times : R_{\mathfrak p}^\times)$ is finite if and only if $R/\mathfrak p$ is finite ([30, Theorem 2.1]).

3. Lemma 4.1 shows that the elasticity of a finitely primary monoid of rank 1 is completely determined by its value semigroup. The interplay of algebraic and arithmetical properties of one-dimensional local Mori domains with properties of their value semigroup has found wide attention in the literature ([5, 4, 10]).

4. For every $d \in \mathbb{N}$, there is a v-noetherian finitely primary monoid H with $\min \Delta(H) = d$. However, even for orders R in algebraic number fields the precise value of $\min \Delta(R_{\mathfrak p})$, $\mathfrak p \in \mathcal P^*$, is known only for some explicit examples (as discussed in [17, Examples 3.7.3]).

To consider the global case, let H is as in Theorem 4.4 with finite v-class group $\mathcal C_v(H)$, and suppose further that every class contains a minimal prime ideal from $\mathcal P$. If H is seminormal or $|G| \ge 3$, then $\min \Delta(H) = 1$ ([23, Theorem 1.1]).

It is a central but far open problem in factorization theory to characterize when a weakly Krull monoid H and when its monoid $\mathcal I_v^*(H)$ of v-invertible v-ideals are transfer Krull monoids resp. transfer Krull monoids of finite type. To begin with the local case, finitely primary monoids are not transfer Krull and the same is true for finite direct products of finitely primary monoids ([21, Theorem 5.6]). These are one of the spare results available so far which indicate that weakly Krull monoids (with the properties of Theorem 4.4) are transfer Krull only in exceptional cases. Clearly, combining results from Section 3 with Theorem 4.4.3 we obtain examples of when the system of sets of lengths of $\mathcal I_v^*(H)$ does not coincide with $\mathcal L(G)$ for any resp. some finite abelian groups G. Clearly, if $\mathcal L\big(\mathcal I_v^*(H)\big) \ne \mathcal L(G)$ for an abelian group G, then $\mathcal I_v^*(H)$ is not transfer Krull over G.

We formulate one such result (others would be possible) as a corollary. But, of course, we are far away from a characterization of when H and the monoid $\mathcal I_v^*(H)$ are transfer Krull resp. of when $\mathcal L(H)$ or $\mathcal L(\mathcal I_v^*(H))$ coincide with $\mathcal L(G)$ for some finite abelian group G (see Section 5 and Problem 5.9 in [21]).

Corollary 4.6. *Let H be a v-noetherian weakly Krull monoid with conductor $\emptyset \ne \mathfrak f = (H : \widehat H) \subsetneq H$ such that $H_{\mathfrak p}$ is finitely primary for each $\mathfrak p \in \mathfrak X(H)$ and $\mathcal I_v^*(H)$ has accepted elasticity. Let $\mathfrak p_1, \ldots, \mathfrak p_s$ be the minimal prime ideals with $\rho\big(H_{\mathfrak p_i}\big) = \rho\big(\mathcal I_v^*(H)\big) > 1$.*

1. *If $\gcd\big(\min \Delta(H_{\mathfrak p_1}), \ldots, \min \Delta(H_{\mathfrak p_s})\big) > 1$ and G is a finite abelian group with $\mathcal L\big(\mathcal I_v^*(H)\big) = \mathcal L(G)$, then G is cyclic of order 4, 6, or 10.*
2. *If there is an $i \in [1, s]$ with $\min \Delta(H_{\mathfrak p_i}) > 1$ and G is a finite abelian group with $\mathcal L\big(\mathcal I_v^*(H)\big) = \mathcal L(G)$, then G does not have rank two and is not of the form $C_{p^k}^r$ with $k, r \in \mathbb N$, $r \ge 2$, and p prime with $p^k \ge 3$. Moreover, if Conjecture 3.20 holds true, then G is either cyclic or isomorphic to $C_2^{1+\min \Delta(H_{\mathfrak p_i})}$.*

Proof. 1. We set $d = \gcd\big(\min \Delta(H_{\mathfrak p_1}), \ldots, \min \Delta(H_{\mathfrak p_s})\big)$. Then Theorem 4.4.3 and Lemma 4.3.2 imply that $\min \Delta_\rho\big(\mathcal I_v^*(H)\big) = d$. Thus the assertion follows from Theorem 3.5.

2. We set $\mathfrak p = \mathfrak p_i$, $\min \Delta(H_p) = d$, and let G be a finite abelian group such that $\mathcal L(G) = \mathcal L\big(\mathcal I_v^*(H)\big)$. Then Theorem 4.4.3 implies that $d \in \Delta_\rho^*\big((\mathcal I_v^*(H)\big) \subset \Delta_\rho\big((\mathcal I_v^*(H)\big) = \Delta_\rho(G)$. Thus the assertion follows from Theorems 3.7, 3.11, 3.13 and Conjecture 3.20. □

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