On the Erdős-Ginzburg-Ziv constant of groups of the form $C_2^r \oplus C_n$

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Abstract

Let G be a finite abelian group. The Erdős-Ginzburg-Ziv constant $\mathsf{s}(G)$ of G is defined as the smallest integer $l \in \mathbb{N}$ such that every sequence S over G of length $|S| \ge l$ has a zero-sum subsequence T of length $|T| = \exp(G)$. The value of this classical invariant for groups with rank at most two is known. But the precise value of $\mathsf{s}(G)$ for the groups of rank larger than two is difficult to determine. In this paper we pay our attentions to the groups of the form $C_2^{r-1} \oplus C_{2n}$, where $r \ge 3$ and $n \ge 2$. We give a new upper bound of $\mathsf{s}(C_2^{r-1} \oplus C_{2n})$ for odd integer n. For $r \in [3, 4]$, we obtain that $\mathsf{s}(C_2^2 \oplus C_{2n}) = 4n + 3$ for $n \ge 2$ and $\mathsf{s}(C_2^3 \oplus C_{2n}) = 4n + 5$ for $n \ge 36$.

Key Words: zero-sum sequence, short zero-sum sequence, EGZ constant, Davenport constant.

1 Introduction

Let G be a finite abelian group. We define some central invariants in zero-sum theory which have been studied since the 1960s. Let

- $\mathsf{D}(G)$ denote the smallest integer $l \in \mathbb{N}$ such that every sequence S over G of length $|S| \ge l$ has a nonempty zero-sum subsequence.
- $\mathsf{s}(G)$ denote the smallest integer $l \in \mathbb{N}$ such that every sequence S over G of length $|S| \ge l$ has a zero-sum subsequence T of length $|T| = \exp(G)$.
- $\eta(G)$ denote the smallest integer $l \in \mathbb{N}$ such that every sequence S over G of length $|S| \ge l$ has a nonempty zero-sum subsequence T of length $|T| \in [1, \exp(G)]$.

$\mathsf{D}(G)$ is called the *Davenport constant* of G and $\mathsf{s}(G)$ the *Erdős-Ginzburg-Ziv* (EGZ) constant of G. For the historical development of the field and the contributions of many authors we refer to the surveys [8, 11]. Here we can only provide a brief summary. There is the following chain of inequalities ([12, 5.7.2 and 5.7.4])

$$\mathsf{D}(G) \le \eta(G) \le \mathsf{s}(G) - \exp(G) + 1 \le |G| \,. \tag{1.1}$$

Clearly, equality holds throughout for cyclic groups which implies the classical Theorem of Erdős-Ginzburg-Ziv dating back to 1961 and stating that $\mathsf{s}(G) = 2|G| - 1$ ([4]). Since about ten years the precise value of all three invariants is known for groups having rank at most two. We have ([12, Theorem 5.8.3])

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Theorem A. *Let* $G = C_{n_1} \oplus C_{n_2}$ *with* $1 \le n_1 \mid n_2$*. Then*

$$\mathsf{s}(G) = 2n_1 + 2n_2 - 3\,, \quad \eta(G) = 2n_1 + n_2 - 2\,, \quad \text{and} \quad \mathsf{D}(G) = n_1 + n_2 - 1\,.$$

In groups of higher rank precise values for any of the three invariants are known only in very special cases. We briefly sketch the state of knowledge with a focus on groups of the form $C_2^r \oplus C_n$, where $r, n \in \mathbb{N}$, which have found special attention in all these investigations.

To begin with the Davenport constant, suppose that $G \cong C_{n_1} \oplus \ldots \oplus C_{n_r}$, where $1 < n_1 \mid \ldots\ldots \mid n_r$, and set $\mathsf{D}^*(G) = 1 + \sum_{i=1}^r (n_i - 1)$. An example shows that $\mathsf{D}^*(G) \le \mathsf{D}(G)$, and equality holds for p-groups. It is open whether or not equality holds for groups of rank three (for recent progress see [1]). Not even the special case where $G = C_n^3$ is known, but we know that $\mathsf{D}^*(C_2 \oplus C_{2n}^4) < \mathsf{D}(C_2 \oplus C_{2n}^4)$ [13, Theorem 3.1]. If $n \ge 3$ is odd and $r \in \mathbb{N}$, then $\mathsf{D}^*(C_2^r \oplus C_{2n}) = \mathsf{D}(C_2^r \oplus C_{2n})$ if and only if $r \le 3$. If $r \le 2$, then the structure of the minimal zero-sum sequences over $C_2^3 \oplus C_{2n}$ is well-known ([15]). The only groups G with $\mathsf{D}^*(G) < \mathsf{D}(G)$, for which the precise value of $\mathsf{D}(G)$ is known are groups of the form $G = C_2^4 \oplus C_{2n}$ for odd $n \ge 70$ ([2, Theorem 5.8]).

A simple example shows that $\eta(G) \le \mathsf{s}(G) - \exp(G) + 1$, and the standing conjecture (due to Weidong Gao [7]) states that equality holds for all groups G. This has been confirmed for a variety of groups (see [6] without knowing the precise value of $\eta(G)$ or $\mathsf{s}(G)$). If G is an elementary 2-group, then it can be seen right from the definitions that $\eta(G) = |G|$ and that $\mathsf{s}(G) = |G| + 1$. If $G = C_3^r$, then $(\mathsf{s}(G) - 1)/2$ is the maximal size of a cap in the r-dimensional affine space over $\mathbb{F}_3$ ([3, Lemma 5.2]). This invariant has been studied in finite geometry since decades, but precise values are known only for $r \le 6$ ([14]). For arbitrary primes p, the precise values of $\eta(C_p^3)$ and $\mathsf{s}(C_p^3)$ are unknown. However, there are standing conjectures which have been verified in very special cases (see [10]), and also the structure of sequences of length $\mathsf{D}(G) - 1$ (resp. $\eta(G) - 1$ or $\mathsf{s}(G) - 1$) that do not have a zero-sum subsequence (of the required length) has been studied for groups of the form C_n^r ([9, Theorem 3.2]). For recent precise results for $\eta(G)$ and $\mathsf{s}(G)$ we refer the reader to [16] and to [5].

In the present paper we focus on the EGZ constant $\mathsf{s}(G)$ and on $\eta(G)$ for groups of the form $C_2^{r-1} \oplus C_{2n}$, where $n \ge 2$ is an integer. Our first result provides the best upper bound on $\mathsf{s}(C_2^r \oplus C_n)$ for $n \ge 3$ odd, which is known so far.

Theorem 1.1. $\mathsf{s}(C_2^{r-1} \oplus C_{2n}) \le 4n + 2^r - 5$ *where* $r \ge 3$ *is a positive integer and* $n \ge 3$ *is an odd integer.*

If $G = C_2^{r-1} \oplus C_{2n}$ and $r \in [3, 4]$, then we can provide precise results.

Theorem 1.2. *Let* $n \ge 2$*.*

1. $\eta(C_2^2 \oplus C_{2n}) = 2n + 4$ *and* $\mathsf{s}(C_2^2 \oplus C_{2n}) = 4n + 3$.
2. $\eta(C_2^3 \oplus C_{2n}) = 2n + 6$, *and if* $n \ge 36$*, then* $\mathsf{s}(C_2^3 \oplus C_{2n}) = 4n + 5$.

2 Notations and Terminology

Our notations and terminology are consistent with [8] and [11]. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{P} \subseteq \mathbb{N}$ the set of prime numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For real numbers $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. Throughout this paper, all abelian groups will be written additively, and for $n \in \mathbb{N}$, we denote by C_n a cyclic group with n elements.

Let G be a finite abelian group. We know that $|G| = 1$ or $G \cong C_{n_1} \oplus \cdots \oplus C_{n_r}$ with $1 < n_1 \mid \cdots \mid n_r \in \mathbb{N}$, where $r = \mathsf{r}(G) \in \mathbb{N}$ is the *rank* of G and $n_r = \exp(G)$ is the *exponent* of G. We denote $|G|$ the *cardinality* of G, and $\mathsf{ord}(g)$ the *order* of elements $g \in G$. For convenience, denote $C_n^r = C_{n_1} \oplus \cdots \oplus C_{n_r}$ if $n_1 = \cdots = n_r = n \in \mathbb{N}$.

Let $\mathscr{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathscr{F}(G)$ are called *sequences* over G. A sequence $S \in \mathscr{F}(G)$ will be written in the form

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)}$$

with $\mathsf{v}_g(S) \in \mathbb{N}_0$ for all $g \in G$. We call $\mathsf{v}_g(S)$ the multiplicity of g in S, and if $\mathsf{v}_g(S) > 0$ we say that S contains g. If for all $g \in G$ we have $\mathsf{v}_g(S) = 0$, then we call S the *empty sequence* and denote $S = 1 \in \mathscr{F}(G)$. A sequence S is called squarefree if $\mathsf{v}_g(S) \le 1$ for all $g \in G$. Apparently, a squarefree sequence over G can be considered as a subset of G. Let $g_0 \in G$, we denote

$$g_0 + S = (g_0 + g_1) \cdot \ldots \cdot (g_0 + g_l).$$

A sequence $S_1 \in \mathscr{F}(G)$ is called a subsequence of S if $\mathsf{v}_g(S_1) \le \mathsf{v}_g(S)$ for all $g \in G$, and denoted by $S_1 \mid S$. If $S_1 \mid S$, we denote

$$S \cdot S_1^{-1} = \prod_{g \in G} g^{\mathsf{v}_g(S) - \mathsf{v}_g(S_1)} \in \mathscr{F}(G).$$

If S_1 is not a subsequence of S, we write $S_1 \nmid S$. Let $S_1, S_2 \in \mathscr{F}(G)$, we set

$$S_1 \cdot S_2 = \prod_{g \in G} g^{\mathsf{v}_g(S_1) + \mathsf{v}_g(S_2)} \in \mathscr{F}(G).$$

Furthermore, we call $S_1, \ldots, S_t$ $(t \ge 2)$ are disjoint subsequences of S, if $S_1 \cdot \ldots \cdot S_t \mid S$.

For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\mathsf{v}_g(S)} \in \mathscr{F}(G),$$

we list the following definitions

$|S| = l = \sum_{g \in G} \mathsf{v}_g(S) \in \mathbb{N}_0$ the *length* of S,

$\mathsf{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\} \subseteq G$ the *support* of S,

$\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g \in G} \mathsf{v}_g(S) g \in G$ the *sum* of S,

$\sum(S) = \{\sum_{i \in I} g_i \mid I \subseteq [1, l] \text{ with } 1 \le |I| \le l\}$ the set of all *subsums* of S,

The sequence S is called

- *zero-sum free* if $0 \notin \sum(S)$,
- a *zero-sum sequence* if $\sigma(S) = 0$,
- a *short zero-sum sequence* if $\sigma(S) = 0$ and $|S| \in [1, \exp(G)]$.

Every map of abelian groups $\phi : G \to H$ extends to a map from the sequences over G to the sequences over H by setting $\phi(S) = \phi(g_1) \cdot \ldots \cdot \phi(g_l)$. If ϕ is a homomorphism, then $\phi(S)$ is a zero-sum sequence if and only if $\sigma(S) \in \mathsf{ker}(\phi)$.

Let $G = H \oplus K$ be a finite abelian group. Let ϕ denote the projection from G to H and ψ denote the projection from G to K. If $S \in \mathscr{F}(G)$ such that $\sigma(\phi(S)) = 0$, then $\sigma(S) = \sigma(\psi(S)) \in \ker(\phi) = K$.

Lemma 2.1. *Let G be a cyclic group of order $n \ge 2$.*

1. *A sequence $S \in \mathscr{F}(G)$ is zero-sum free of length $|S| = n-1$ if and only if $S = g^{n-1}$ for some $g \in G$ with $\mathsf{ord}(g) = n$.*

2. *Let S be a zero-sum free sequence over G with length greater than $n/2$. Then there exists an element $g \in G$ of order n such that*
$$S = (k_1 g) \cdot \ldots \cdot (k_{|S|} g),$$
where $1 \le k_1 \le \cdots \le k_{|S|}$, $k = k_1 + \cdots + k_{|S|} < n$, and $\Sigma(S) = \{g, 2g, \ldots, kg\}$.

3. *Let $S \in \mathscr{F}(G)$ a sequence of length $|S| = \mathsf{s}(G) - 1$. Then the following statements are equivalent:*
   (a) S has no zero-sum subsequence of length n.
   (b) $S = (gh)^{n-1}$ where $g, h \in G$ with $\mathsf{ord}(g-h) = n$.

Proof. See [11, Cor. 2.1.4, Th.5.1.8, Prop.5.1.12]. □

Lemma 2.2. *([6, Theorem 1.2]) Let $G = H \oplus C_{mn}$ be a finite abelian group where $H \subseteq G$ is a subgroup with $\exp(H) = m \ge 2$ and $n \in \mathbf{N}$. If $n \ge \max\{m|H|+1, 4|H|+2m\}$, then $\mathsf{s}(G) = \eta(G) + \exp(G) - 1$.*

Lemma 2.3. *Let $G = H \oplus C_n$ be a finite abelian group where $H \subseteq G$ is a subgroup with $\exp(H) \mid \exp(G) = n$. Then*
$$\eta(G) \ge 2(\mathsf{D}(H) - 1) + n \quad \text{and} \quad s(G) \ge 2(\mathsf{D}(H) - 1) + 2n - 1\,.$$
In particular, we have for all $r, n \in \mathbb{N}$,
$$\eta(C_2^{r-1} \oplus C_{2n}) \ge 2n + 2r - 2 \quad \text{and} \quad \mathsf{s}(C_2^{r-1} \oplus C_{2n}) \ge 4n + 2r - 3\,.$$

Proof. The main statement follows from [3, Lemma 3.2]. If $H = C_2^{r-1}$, then $\mathsf{D}(H) = r$ and the second statement follows. □

Lemma 2.4. *Let W be a squarefree sequence of length $|W| \ge 9$ over $C_2^4 \setminus \{0\}$. Then for any element $w \mid W$, there exist $|W|-8$ disjoint subsequences $R_1, \ldots, R_{|W|-8}$ of Ww^{-1} such that $\sigma(R_i) = w$ and $|R_i| = 2$ for all $i \in [1, |W|-8]$. In particular, W has at least $|W|-8$ distinct zero-sum subsequences of length 3 containing w.*

Proof. For any element $w \mid W$, we can always find a set $A = \{a_1, \ldots, a_7\} \subseteq C_2^4 \setminus \{0\}$ satisfying $C_2^4 \setminus \{0\} = \{w\} \cup A \cup w + A$. Let $T_i = a_i \cdot (w + a_i)$ for each $i \in [1, 7]$. Therefore $w = \sigma(T_1) = \ldots = \sigma(T_7)$ and $\mathsf{supp}(wT_1 \cdot \ldots \cdot T_7) = C_2^4 \setminus \{0\}$.

Since
$$\begin{aligned}&\left|\left\{T_i \;\middle|\; i \in [1,7] \text{ and } T_i \nmid W\right\}\right| \le \left|\left\{g \in C_2^4 \setminus \{0\} \;\middle|\; g \mid T_i \text{ for some } i \in [1,7] \text{ and } g \nmid W\right\}\right| \\ \le& |C_2^4 \setminus (\{0\} \cup \mathrm{supp}(W))| = 15 - |W|,\end{aligned}$$
we have that $\left|\left\{T_i \;\middle|\; i \in [1,7] \text{ and } T_i \mid W\right\}\right| \ge 7 - (15 - |W|) = |W| - 8 \ge 1$. In particular, for each $i \in [1, 7]$, if $T_i \mid W$, then wT_i is a zero-sum subsequence of W. □

3 The proof of Theorem 1.1 and Theorem 1.2.1

Lemma 3.1. *Let $G = H \oplus K$ be a finite abelian group, where $H \cong C_2^r$ with $r \geq 3$ a positive integer and $K \cong C_n$ with $n \geq 3$ an odd integer. Denote ϕ_r to be the projection from G to H and ψ_r to be the projection from G to K.*

Let S_r be a sequence over $C_2^r \oplus C_n$ such that $\phi_r(S_r)$ is a squarefree sequence with $\mathsf{supp}(\phi_r(S_r)) = H \setminus \{0\}$. If the following property **P1** *holds,*

P1. *For any two distinct subsequences T_1, T_2 of S_r with $|T_1| = |T_2| = 4$ and $\phi_r(\sigma(T_1)) = \phi_r(\sigma(T_2)) = 0$, we have that $\sigma(T_1) = \sigma(T_2)$.*

then $|\mathsf{supp}(\psi_r(S_r))| = 1$.

Proof. We proceed by induction on r.

Suppose that $r = 3$. Let (e_1, e_2, e_3) be a basis of H, and e be a basis of K.

Since $\phi_3(S_3)$ is a squarefree sequence with $\mathsf{supp}(\phi_3(S_3)) = H \setminus \{0\}$, we can assume $S_3 = g_1 g_2 g_3 g_4 g_5 g_6 g_7$, where

$$g_1 = e_1 + a_1 e, \quad g_2 = e_2 + e_3 + a_2 e, \quad g_3 = e_2 + a_3 e, \quad g_4 = e_1 + e_3 + a_4 e,$$
$$g_5 = e_3 + a_5 e, \quad g_6 = e_1 + e_2 + a_6 e, \quad g_7 = e_1 + e_2 + e_3 + a_7 e, \text{ and } a_1, \ldots, a_7 \in [0, n-1].$$

By the property **P1** and

$$\begin{aligned} &\phi_3(g_1 + g_3 + g_5 + g_7) = \phi_3(g_1 + g_4 + g_6 + g_7) = \phi_3(g_2 + g_3 + g_6 + g_7) = \phi_3(g_2 + g_4 + g_5 + g_7) \\ =&\phi_3(g_1 + g_2 + g_3 + g_4) = \phi_3(g_1 + g_2 + g_5 + g_6) = \phi_3(g_3 + g_4 + g_5 + g_6) = 0, \end{aligned}$$

we have that

$$\begin{aligned} g_1 + g_3 + g_5 + g_7 &= g_1 + g_4 + g_6 + g_7 = g_2 + g_3 + g_6 + g_7 = g_2 + g_4 + g_5 + g_7 \\ &= g_1 + g_2 + g_3 + g_4 = g_1 + g_2 + g_5 + g_6 = g_3 + g_4 + g_5 + g_6. \end{aligned}$$

By easily calculation, we obtain that $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7$, which implies that $|\mathsf{supp}(\psi_3(S_3))| = 1$.

Suppose that the conclusion is correct for $r = d \geq 3$. We want to prove the conclusion is also correct for $r = d + 1$.

Let $(e_1, \ldots, e_{d+1})$ be a basis of H and e be a basis of K. Denote by $A_i = \langle e_1, \ldots, e_i \rangle \setminus \{0\}$, where $i \in [1, d+1]$. Then $A_{d+1} = A_d \cup (e_{d+1} + A_d) \cup \{e_{d+1}\}$. Since $\phi_{d+1}(S_{d+1})$ is a squarefree sequence with $\mathsf{supp}(\phi_{d+1}(S_{d+1})) = H \setminus \{0\}$, we can assume $S_{d+1} = \prod_{u \in A_{d+1}} u + c_u e$, where $c_u \in [0, n-1]$ for each $u \in A_{d+1}$.

Let

$$W_1 = \prod_{u \in A_d} u + c_u e, \quad W_2 = \prod_{u \in e_{d+1} + A_d} u - e_{d+1} + c_u e, \quad H' = \langle e_1, \ldots, e_d \rangle, \text{ and } G' = H' \oplus K.$$

Then W_1, W_2 are sequences over G' and $S_{d+1} = W_1 \cdot (e_{d+1} + W_2) \cdot (e_{d+1} + c_{e_{d+1}} e)$. Since $\mathsf{r}(H') = d$, $\phi_{d+1}\big|_{G'}$ is the projection from G' to H', and $\psi_{d+1}\big|_{G'}$ is the projection from G' to K, we obtain that W_1, W_2 satisfy the property **P1** for $r = d$ which implies that $|\operatorname{supp}(\psi_{d+1}(W_1))| = |\operatorname{supp}(\psi_{d+1}(W_2))| = 1$. Therefore we can assume that $c_u = x$ for all $u \in A_d$ and $c_u = y$ for all $u \in e_{d+1} + A_d$ where $x, y \in [0, n-1]$.

Let $B = \{0, e_1, e_2, e_1 + e_2\}$ and $T_1 = \prod_{u \in e_3 + B} u + c_u e$, $T_2 = \prod_{u \in e_{d+1} + e_3 + B} u + c_u e$, $T_3 = \prod_{u \in e_{d+1} + B} u + c_u e$. Then $T_1 T_2 T_3 \mid S_{d+1}$ and $\phi_{d+1}(\sigma(T_1)) = \phi_{d+1}(\sigma(T_2)) = \phi_{d+1}(\sigma(T_3)) = 0$. By the property **P1** and $|T_1| = |T_2| = |T_3| = 4$, we obtain that $\psi_{d+1}(\sigma(T_1)) = \psi_{d+1}(\sigma(T_2)) = \psi_{d+1}(\sigma(T_3))$ and hence

$$4x \equiv 4y \equiv 3y + c_{e_{d+1}} \pmod{n}.$$

It follows that $x = y = c_{e_{d+1}}$ which implies that $|\operatorname{supp}(\psi(S_{d+1}))| = 1$. $\square$

Proof of Theorem 1.1. Let $G = H \oplus K$ be a finite abelian group, where $H \cong C_2^r$ with $r \geq 3$ a positive integer and $K \cong C_n$ with $n \geq 3$ an odd integer. Denote ϕ to be the projection from G to H and ψ to be the projection from G to K.

Let S be a sequence over G with $|S| = 4n + 2^r - 5$. Assume to the contrary that S contains no zero-sum subsequence of length $2n$.

Suppose that $\phi(G) = H = \{h_0, h_1, \ldots, h_{2^r-1}\}$. We can assume that

$$\phi(S) = h_0^{n_0} \cdot \ldots \cdot h_{2^r-1}^{n_{2^r-1}} \text{ and } S = W_0 \cdot \ldots \cdot W_{2^r-1}$$

where $n_0, \ldots, n_{2^r-1} \in \mathbb{N}_0$, and $\phi(W_i) = h_i^{n_i}$ for all $i \in [0, 2^r - 1]$.

Then S allows a product decomposition $S = S_1 \cdot \ldots \cdot S_k \cdot S_0$ satisfying that $\phi(S_0)$ is squarefree, and for each $i \in [1, k]$, $|S_i| = 2$ and $\sigma(\phi(S_i)) = 0$. Therefore $S_i \mid W_j$ for some $j \in [0, 2^r - 1]$.

Since $4n + 2^r - 5 = |S| = 2k + |S_0| \leq 2k + 2^r$, we obtain that $k \geq 2n - 2$. By our assumption, $\sigma(S_1) \cdot \ldots \cdot \sigma(S_k) \in \mathscr{F}(\ker(\phi))$ has no subsequence of length n. Therefore by Lemma 2.1.3, we have that $k = 2n - 2$, $|S_0| = 2^r - 1$ and

$$\sigma(S_1) \cdot \ldots \cdot \sigma(S_{2n-2}) = g^{n-1} {g_1}^{n-1},$$

where $g, g_1 \in \mathsf{ker}(\phi)$ and $\mathsf{ord}(g - g_1) = n$.

Since $|\operatorname{supp}(\phi(S_0))| = |S_0| = 2^r - 1$, let $\{b\} = \phi(G) \setminus \operatorname{supp}(\phi(S_0))$ and

$$S' = S + b - \frac{n+1}{2} g_1 = (S_1 + b - \frac{n+1}{2} g_1) \cdot \ldots \cdot (S_{2n-2} + b - \frac{n+1}{2} g_1) \cdot (S_0 + b - \frac{n+1}{2} g_1).$$

Therefore S' has no zero-sum subsequence of length $2n$ and $0 \notin \operatorname{supp}(\phi(S_0 + b)) = \operatorname{supp}(\phi(S_0 + b - \frac{n+1}{2} g_1))$,

$$\sigma(S_1 + b - \frac{n+1}{2} g_1) \cdot \ldots \cdot \sigma(S_{2n-2} + b - \frac{n+1}{2} g_1) = (g - g_1)^{n-1} \cdot 0^{n-1}.$$

Then without loss of generality, we can assume that $0 \notin \operatorname{supp}(\phi(S_0))$ and

$$\sigma(S_1) = \cdots = \sigma(S_{n-1}) = g_1 = 0 \text{ and } \sigma(S_n) = \cdots = \sigma(S_{2n-2}) = g.$$

Case 1. There exists $T \mid S_0$ such that $|T| = 4$, $\sigma(\phi(T)) = 0$ and $\sigma(\psi(T)) \neq g$.

Then $\psi(\sigma(T)) = tg$ where $t \in [2, n]$. By calculation we get

$$\sigma(S_1 \cdot \ldots \cdot S_{t-2} \cdot S_n \cdot \ldots \cdot S_{2n-1-t} \cdot T) = 0, \quad \text{and}$$

$$|S_1 \cdot \ldots \cdot S_{t-2} \cdot S_n \cdot \ldots \cdot S_{2n-1-t} \cdot T| = 2(t-2) + 2(n-t) + 4 = 2n,$$

a contradiction.

Case 2. For any subsequence $T \mid S_0$ satisfying $|T| = 4$ and $\phi(\sigma(T)) = 0$, we have $\psi(\sigma(T)) = g$.

By $\mathsf{supp}(\phi(S_0)) = \phi(G) \setminus \{0\}$ and Lemma 3.1, we have $\mathsf{supp}(\psi(S_0)) = \{a\}$ for some $a \in \psi(G)$. Let T be a subsequence of S_0 satisfying $|T| = 4$ and $\phi(\sigma(T)) = 0$, then $\psi(\sigma(T)) = 4a = g$ which implies that

$$a = \{\frac{n+1}{4} g\} \quad \text{if } n \equiv 3 \pmod 4 \qquad \text{and} \qquad a = \{\frac{3n+1}{4} g\} \quad \text{if } n \equiv 1 \pmod 4. \tag{3.1}$$

Without loss of generality, we can assume $h_0 = 0 \notin \operatorname{supp}(\phi(S_0))$ and $|W_1| \geq \cdots \geq |W_{2^r-1}|$. Then $2 \nmid |W_i|$ for all $i \in [1, 2^r - 1]$.

We can distinguish the following two cases.

Subcase 2.1. $|W_1| \geq 3$.

Without loss of generality, we can assume that $a_1a_2a_0 \mid W_1$ and $S_i = a_1a_2$, $a_0 \mid S_0$, where $i \in [1, 2n-2]$. Then $\sigma(\psi(S_i)) = \psi(a_1) + \psi(a_2) \in \{0, g\}$ and hence $\psi(a_1) \neq \psi(a_0)$ or $\psi(a_2) \neq \psi(a_0)$ by Equation (3.1). Without loss of generality, we can assume that $\psi(a_1) \neq \psi(a_0)$. Let $S_i' = a_0a_2$ and $S_0' = S_0a_0^{-1}a_1$. Since $\sigma(\phi(S_1)) \cdot \ldots \cdot \sigma(S_{i-1}) \cdot \sigma(S_i') \cdot \sigma(S_{i+1}) \cdot \ldots \cdot \sigma(S_{2n-2})$ has no subsequence of length n, by Lemma 2.1.3 we obtain that $\sigma(\phi(S_i')) = \sigma(\phi(S_i))$ which implies that $\psi(a_1) = \psi(a_0)$, a contradiction.

Subcase 2.2. $|W_1| \leq 2$.

Since $2 \geq |W_1| \geq \cdots \geq |W_{2^r-1}|$ and $2 \nmid |W_i|$ for all $i \in [1, 2^r-1]$, we have that $|W_1| = \cdots = |W_{2^r-1}| = 1$. Then

$$|W_0| = |S| - \sum_{i=1}^{2^r-1} |W_i| = 4n + 2^r - 5 - (2^r - 1) = 4n - 4 \geq 3n - 1.$$

Since W_0 is a sequence over $\ker(\phi) \cong C_n$, there are two disjoint zero-sum subsequences V_1, V_2 of length n by $s(C_n) = 2n - 1$(see Theorem **A**). Therefore V_1V_2 is a zero-sum subsequence of length $2n$, a contradiction. □

Proof of Theorem 1.2.1. By Lemma 2.3 and Inequality 1.1, we only need to prove that $s(G) \leq 4n + 3$. If n is odd, it follows immediately by Theorem 1.1. Thus we can always assume that n is even.

Let (e_1, e_2, e) be a basis of $G = C_2^2 \oplus C_{2n}$ with $\mathrm{ord}(e_1) = \mathrm{ord}(e_2) = 2$ and $\mathrm{ord}(e) = 2n$ and S be any sequence over G with $|S| = 4n + 3$. Assume to the contrary that S contains no zero-sum subsequence of length $2n$.

Let $\theta : G \to G$ be the homomorphism defined by $\theta(e_1) = e_1$, $\theta(e_2) = e_2$ and $\theta(e) = ne$. Then $\mathsf{ker}(\theta) = \langle 2e \rangle \cong C_n$ and $\theta(G) = \langle e_1, e_2, ne \rangle \cong C_2^3$.

Let $\theta(G) = \{h_0, h_1, \ldots, h_7\}$. We can assume that

$$\theta(S) = h_0^{n_0} \cdot \ldots \cdot h_7^{n_7} \text{ and } S = W_0 \cdot \ldots \cdot W_7$$

where $n_0, \ldots, n_7 \in \mathbb{N}_0$, and $\theta(W_i) = h_i^{n_i}$ for all $i \in [0, 7]$.

Then S allows a product decomposition $S = S_1 \cdot \ldots \cdot S_k \cdot S_0$ satisfying that $\theta(S_0)$ is squarefree, and for each $i \in [1, k]$, $|S_i| = 2$ and $\sigma(\theta(S_i)) = 0$. Therefore $S_i \mid W_j$ for some $j \in [0, 7]$.

Since $4n+3 = |S| = 2k+|S_0| \leq 2k+8$, we obtain that $k \geq 2n-2$. By our assumption, $\sigma(S_1) \cdot \ldots \cdot \sigma(S_k) \in \mathscr{F}(\ker(\theta))$ has no subsequence of length n. Therefore by Lemma 2.1.3, we have that $k = 2n - 2$, $|S_0| = 7$ and

$$\sigma(S_1) \cdot \ldots \cdot \sigma(S_{2n-2}) = (2ke)^{n-1}(2k_1e)^{n-1},$$

where $k, k_1 \in [0, n-1]$ and $\gcd(k - k_1, n) = 1$. Since n is even, without loss of generality, we can assume that k_1 is even.

Since $|\operatorname{supp}(\theta(S_0))| = |S_0| = 7$, we let $\{b\} = \theta(G) \setminus \operatorname{supp}(\theta(S_0)) \in G$ and let

$$S' = S + b - k_1e = (S_1 + b - k_1e) \cdot \ldots \cdot (S_{2n-2} + b - k_1e) \cdot (S_0 + b - k_1e).$$

Therefore S' has no zero-sum subsequence of length $2n$ and $0 \notin \operatorname{supp}(\theta(S_0 + b)) = \operatorname{supp}(\theta(S_0 + b - k_1e))$,

$$\sigma(S_1 + b - k_1e) \cdot \ldots \cdot \sigma(S_{2n-2} + b - k_1e) = (2(k - k_1)e)^{n-1} \cdot 0^{n-1}.$$

Then without loss of generality, we can assume that $0 \notin \operatorname{supp}(\theta(S_0))$ and

$$\sigma(S_1) = \cdots = \sigma(S_{n-1}) = 2k_1e = 0 \text{ and } \sigma(S_n) = \cdots = \sigma(S_{2n-2}) = 2ke = 2e.$$

Case 1. There exists $T \mid S_0$ such that $|T| = 4$, $\theta(\sigma(T)) = 0$ and $\sigma(T) \neq 2e$.

Then $\sigma(T) = 2te$ where $t \in [2, n]$. By calculation we get

$$\sigma(S_1 \cdot \ldots \cdot S_{t-2} \cdot S_n \cdot \ldots \cdot S_{2n-1-t} \cdot T) = 0 \quad \text{and}$$

$$|S_1 \cdot \ldots \cdot S_{t-2} \cdot S_n \cdot \ldots \cdot S_{2n-1-t} \cdot T| = 2(t-2) + 2(n-t) + 4 = 2n,$$

a contradiction.

Case 2. For any subsequence $T \mid S_0$ satisfying $|T| = 4$ and $\theta(\sigma(T)) = 0$, we have $\sigma(T) = 2e$.

Since $\mathsf{supp}(\theta(S_0)) = \theta(G) \setminus \{0\}$, we can assume $S_0 = g_1g_2g_3g_4g_5g_6g_7$, where

$$g_1 = e_1 + 2a_1e, \quad g_2 = e_2 + e + 2a_2e, \quad g_3 = e_2 + 2a_3e, \quad g_4 = e_1 + e + 2a_4e,$$
$$g_5 = e + 2a_5e, \quad g_6 = e_1 + e_2 + 2a_6e, \quad g_7 = e_1 + e_2 + e + 2a_7e, \text{ and } a_1, \ldots, a_7 \in [0, n-1].$$

Since $\theta(g_1+g_3+g_5+g_7) = \theta(g_2+g_4+g_5+g_7) = \theta(g_1+g_2+g_3+g_4)$, we obtain the following equations

$$g_1 + g_3 + g_5 + g_7 = g_2 + g_4 + g_5 + g_7 = g_1 + g_2 + g_3 + g_4 = 2e,$$

which implies that

$$2e + 2(a_1 + a_3 + a_5 + a_7)e = 4e + 2(a_2 + a_4 + a_5 + a_7)e = 2e + 2(a_1 + a_2 + a_3 + a_4)e = 2e.$$

Therefore

$$a_1 + a_3 + a_5 + a_7 \equiv 0 \pmod{n},$$
$$a_2 + a_4 + a_5 + a_7 \equiv -1 \pmod{n},$$
$$a_1 + a_2 + a_3 + a_4 \equiv 0 \pmod{n}.$$

Thus $2(a_1+a_3)+a_2+a_4+a_5+a_7 \equiv 0 \pmod{n}$, which implies that $2(a_1+a_3) \equiv 1 \pmod{n}$, a contradiction to n is even. □

4 Preparatory results about $C_2^3 \oplus C_{2n}$

In the whole section, we consider the group $G = C_2^3 \oplus C_{2n}$, where $n \geq 3$ is an odd integer. Thus $G \cong C_2^4 \oplus C_n$. Let $G = H \oplus K$, where H, K are subgroups of G with $H \cong C_2^4$ and $K \cong C_n$. Denote ϕ to be the projection from G to H and ψ to be the projection from G to K.

Lemma 4.1. *Let G, H, K and ϕ, ψ be as above. If S is a sequence of length 10 over G such that $\phi(S)$ is a squarefree sequence over $H \setminus \{0\}$, then S has two distinct subsequences T_1 and T_2 of length $\{|T_1|, |T_2|\} \subseteq [3, 4]$ satisfying $\sigma(\phi(T_1)) = \sigma(\phi(T_2)) = 0$ but $\sigma(\psi(T_1)) \neq \sigma(\psi(T_2))$.*

Proof. By Lemma 2.4, S has at least two distinct subsequences W_1, W_2 of length 3 such that $\phi(\sigma(W_1)) = \phi(\sigma(W_2)) = 0$.

Assume to the contrary that for any zero-sum subsequence $\phi(T)$ of $\phi(S)$ with length 3 or 4, we have $\sigma(\psi(T)) = e$, where $e \in K \setminus \{0\}$.

By Lemma 2.4, there exists a subsequence T of S such that $|T| = 3$ and $\sigma(\phi(T)) = 0$. Then $\sigma(\psi(T)) = e$ and hence there exists an element $u \mid T$ such that $\psi(u) \neq \frac{n+1}{2}e$.

By Lemma 2.4 again, there exist disjoint subsequences R_1, R_2 of Su^{-1} such that $\sigma(\phi(R_1)) = \sigma(\phi(R_2)) = \phi(u)$ and $|R_1| = |R_2| = 2$. Thus $\phi(R_1R_2)$, $\phi(R_1u)$, and $\phi(R_2u)$ are zero-sum sequences which implies that $\sigma(\psi(R_1R_2)) = \sigma(\psi(R_1u)) = \sigma(\psi(R_2u)) = e$. It follows that $\psi(u) = \frac{n+1}{2}e$, a contradiction to the choice of u. □

Lemma 4.2. *Let G, H, K and ϕ, ψ be as above. If $n = 3$ and S is a sequence of length 12 over G such that $\phi(S)$ is a squarefree sequence, then S contains a short zero-sum subsequence.*

Proof. Assume to the contrary that S contains no short zero-sum subsequence. Thus $0 \notin \mathsf{supp}(S)$.

If $0 \in \mathsf{supp}(\phi(S))$, then there exists $g \mid S$ such that $\phi(g) = 0$ and hence $\psi(g) \neq 0$. By $|Sg^{-1}| = 11$ and Lemma 4.1, Sg^{-1} have a subsequence T of length $|T| \in \{3, 4\}$ such that $\sigma(\phi(T)) = 0$ and $\sigma(\psi(T)) \neq \psi(g)$. Since $\sigma(\psi(T)) \neq 0$, we obtain $\sigma(\psi(T)) = 2\psi(g)$ which implies that Tg is a short zero-sum subsequence of S, a contradiction.

Therefore $0 \notin \mathsf{supp}(\phi(S))$. Let $S = g_1 \cdot \ldots \cdot g_{12}$. We distinguish the following four cases to finish the proof.

Case 1. $\mathsf{v}_0(\psi(S)) \geq 5$.

Without loss of generality, we can assume that $\psi(g_1 \cdot \ldots \cdot g_5) = 0^5$. Since $\phi(g_1 \cdot \ldots \cdot g_5) \in \mathscr{F}(\phi(G))$ and $\mathsf{D}(\phi(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $X \mid g_1 \cdot \ldots \cdot g_5$ such that $\sigma(\phi(X)) = 0$ which implies that X is a short zero-sum subsequence of S, a contradiction.

Case 2. $\mathsf{v}_0(\psi(S)) = 4$.

Without loss of generality, we can assume that $\psi(g_1g_2g_3g_4) = 0^4$ and $\psi(g_5g_6g_7g_8) = e^4$ for some $e \in \psi(G) \cong C_3$. Thus $g_1, g_2, g_3, g_4, g_5 + g_6 + g_7, g_5 + g_6 + g_8 \in \phi(G) \cong C_2^4$ and $g_5 + g_6 + g_7 \neq g_5 + g_6 + g_8$. Choose $R = g_5 + g_6 + g_7$ or $g_5 + g_6 + g_8$ such that $\sigma(R) \neq g_1 + g_2 + g_3 + g_4$. Then $g_1g_2g_3g_4\sigma(R)$ has a zero-sum subsequence of length ≤ 4 which implies that S contains a short zero-sum subsequence, a contradiction.

Case 3. $\mathsf{v}_0(\psi(S)) = 3$.

Without loss of generality, we can assume that $\psi(S) = 0^3 \cdot e^u \cdot (2e)^v$, $u+v = 9$, $u \geq v$, and $\psi(g_1g_2g_3) = 0^3$ for some $e \in \psi(G) \cong C_3$.

Suppose that $v = 0$. We assume that $\psi(g_4 \cdot \ldots \cdot g_{12}) = e^9$. Then by Lemma 2.4, $\phi(g_4 \cdot \ldots \cdot g_{12})$ contains a zero-sum subsequence of length 3 which implies that S contains a short zero-sum subsequence of length 3, a contradiction.

Suppose that $v = 1$. We assume that $\psi(g_4 \cdot \ldots \cdot g_{11}) = e^8$ and $\psi(g_{12}) = 2e$. Then $g_1, g_2, g_3, g_{11} + g_{12}, g_4 + g_5 + g_j \in \phi(G) \cong C_2^4$ for any $j \in [6, 10]$. If $\sigma(g_1g_2g_3g_{11}g_{12}) = 0$, then $g_1g_2g_3g_{11}g_{12}$ is a short zero-sum subsequence of S, a contradiction. Thus $\sigma(g_1g_2g_3g_{11}g_{12}) \neq 0$. Since $|[6, 10]| = 5$, there exists an $i \in [6, 10]$ such that

$$g_4 + g_5 + g_i \notin \{\sigma(g_1g_2g_3(g_{11}g_{12})), \sigma(g_1g_2(g_{11}g_{12})), \sigma(g_1g_3(g_{11}g_{12})), \sigma(g_2g_3(g_{11}g_{12}))\}.$$

Therefore $g_1g_2g_3\sigma(g_{11}g_{12})\sigma(g_4g_5g_i)$ contains a zero-sum subsequence of length ≤ 3 which implies that S contains a short zero-sum subsequence, a contradiction.

Suppose that $v \geq 2$. we assume that $\psi(g_4 \cdot \ldots \cdot g_8) = e^5$ and $\psi(g_{11}g_{12}) = (2e)^2$. Then $g_4 + g_{11}, g_5 + g_{12}, g_6 + g_{12} \in \phi(G) \cong C_2^4$ and $g_5 + g_{12} \neq g_6 + g_{12}$. Choose $R = g_5 + g_{12}$ or $g_6 + g_{12}$ such that $\sigma(R) \neq g_4 + g_{11} + g_1 + g_2 + g_3$. Therefore $g_1g_2g_3\sigma(g_4g_{11})\sigma(R)$ contains a zero-sum subsequence of length ≤ 4 which implies that S contains a short zero-sum subsequence, a contradiction.

Case 4. $\mathsf{v}_0(\psi(S)) \leq 2$.

Without loss of generality, we can assume that $\psi(S) = 0^t \cdot e^u \cdot (2e)^v$, $t \in \mathbb{N}_0$, $u + v \geq 10$, and $u \geq v$ for some $e \in \psi(G) \cong C_3$.

Suppose that $u \geq 9$. We assume that $\psi(g_1 \cdot \ldots \cdot g_9) = e^9$. Then by Lemma 2.4, $\phi(g_1 \cdot \ldots \cdot g_9)$ contains a zero-sum subsequence of length 3 which implies that S contains a short zero-sum subsequence of length 3, a contradiction.

Suppose that $u \le 8$. Then $v \ge 2$. We assume that $\psi(g_1 \cdot \ldots \cdot g_u) = e^u$ and $\psi(g_{u+1} \cdot \ldots \cdot g_{u+v}) = (2e)^v$. If there exist $i_1, i_2 \in [1, u]$ and $j_1, j_2 \in [u+1, u+v]$ such that $i_1 \ne i_2$, $j_1 \ne j_2$, and $g_{i_1} + g_{j_1} = g_{i_2} + g_{j_2}$, then $g_{i_1} g_{j_1} g_{i_2} g_{j_2}$ is a short zero-sum subsequence of S, a contradiction. Therefore

$$|\{g_i + g_j \in \phi(G) \mid i \in [1, u] \text{ and } j \in [u+1, u+v]\}| \ge uv \ge v(10-v) \ge 16 = |\phi(G)|.$$

It follows that there exist an $i \in [1, u]$ and a $j \in [u+1, u+v]$ such that $\phi(g_i) = \phi(g_j)$, a contradiction to $\phi(S)$ is squarefree. $\square$

Lemma 4.3. *Let G, H, K and ϕ, ψ be as above. Let $K = \langle e \rangle$ and $S = h_1 \cdot \ldots \cdot h_8$ be a sequence over $G \setminus \{0\}$ with $\phi(h_1) + \phi(h_2) = \phi(h_3) + \phi(h_4) = \phi(h_5) + \phi(h_6) = \phi(h_7) + \phi(h_8)$. If $\phi(S)$ is a squarefree sequence with $0 \notin \operatorname{supp}(\phi(S))$ and satisfies the following property $(*)$:*

$$\begin{cases} \text{For any subsequence } V \text{ of } S \text{ with } \sigma(\phi(V)) = 0, \text{ we have that} \\ \sigma(\psi(V)) = \begin{cases} e, & \text{if } |V| = 3 \text{ or } 4, \\ e \text{ or } 2e, & \text{if } |V| = 5. \end{cases} \end{cases} \qquad (*)$$

then $\operatorname{supp}(\psi(S)) = \{\frac{n+1}{4}e\}$ *if* $n \equiv 3 \pmod 4$ *and* $\operatorname{supp}(\psi(S)) = \{\frac{3n+1}{4}e\}$ *if* $n \equiv 1 \pmod 4$.

Proof. Since $\sigma(\phi(h_1h_2h_3h_4)) = \sigma(\phi(h_3h_4h_5h_6)) = \sigma(\phi(h_5h_6h_1h_2)) = 0$, we obtain that $\sigma(\psi(h_1h_2h_3h_4)) = \sigma(\psi(h_3h_4h_5h_6)) = \sigma(\psi(h_5h_6h_1h_2)) = e$ which implies that $\psi(h_1) + \psi(h_2) = \psi(h_3) + \psi(h_4) = \psi(h_5) + \psi(h_6) = \frac{n+1}{2}e$. With the same reason, we can prove that $\psi(h_7) + \psi(h_8) = \frac{n+1}{2}e$.

Let $\psi(h_i) = k_i e$ where $1 \le i \le 8$ and $0 \le k_i \le n-1$. Without loss of generality, we can assume that $k_1 \le k_2$, $k_3 \le k_4$, $k_5 \le k_6$, $k_7 \le k_8$. We consider the sequence $W = h_1h_2h_3h_5h_7$ (see Figure 4.1).

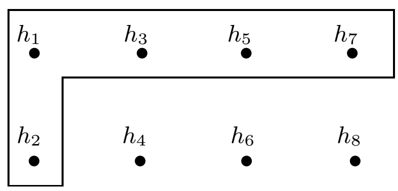

Figure 4.1:

Since $\phi(W) \in \mathscr{F}(\phi(G))$ and $\mathsf{D}(\phi(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $V \mid W$ such that $\sigma(\phi(V)) = 0$ and $|V| \in \{3, 4, 5\}$. We distinguish three cases depending on $|V|$.

Case 1. $|V| = 3$.

Since $0 \notin \operatorname{supp}(\phi(S))$, we obtain that $h_1h_2 \nmid V$. By symmetry, we only need to consider $V = h_1h_3h_5$ or $V = h_2h_3h_5$.

Suppose that $V = h_1h_3h_5$. Then $\sigma(\phi(h_1h_3h_5)) = 0$ and hence $\sigma(\phi(h_1h_4h_6)) = 0$. Thus

$$\sigma(\psi(h_1h_3h_5)) = \sigma(\psi(h_1h_4h_6)) = e$$

which implies that $\sigma(\psi(h_3h_5)) = \sigma(\psi(h_4h_6)) = \frac{n+1}{2}e$ and $\psi(h_1) = \frac{n+1}{2}e$ by $\sigma(\psi(h_3h_4h_5h_6)) = e$. Therefore $\psi(h_2) = 0$, a contradiction to $k_1 \le k_2$.

Suppose that $V = h_2h_3h_5$. Then $\sigma(\phi(h_2h_3h_5)) = 0$ and hence $\sigma(\phi(h_1h_4h_5)) = 0$. Thus

$$\sigma(\psi(h_2h_3h_5)) = \sigma(\psi(h_1h_4h_5)) = e$$

which implies that $\sigma(\psi(h_2h_3)) = \sigma(\psi(h_1h_4)) = \frac{n+1}{2}e$ and $\psi(h_5) = \frac{n+1}{2}e$ by $\sigma(\psi(h_1h_2h_3h_4)) = e$. Therefore $\psi(h_6) = 0$, a contradiction to $k_5 \le k_6$.

Case 2. $|V| = 4$.

Since $\phi(S)$ is squarefree, we obtain that $h_1h_2 \nmid V$. Thus there are only two cases: $V = h_1h_3h_5h_7$ and $V = h_2h_3h_5h_7$.

Suppose that $V = h_1h_3h_5h_7$. Since $\sigma(\phi(h_1h_3h_5h_7)) = \sigma(\phi(h_2h_4h_5h_7)) = 0$, we have that

$$\sigma(\psi(h_1h_3h_5h_7)) = \sigma(\psi(h_2h_4h_5h_7)) = e$$

which implies that $\psi(h_1 + h_3) = \psi(h_2 + h_4)$. By $\psi(h_1 + h_2) = \psi(h_3 + h_4) = \frac{n+1}{2}e$ and $k_1 \le k_2$, $k_3 \le k_4$, we obtain that

$$\psi(h_1) = \psi(h_3) = \psi(h_2) = \psi(h_4) = \frac{n+1}{4}e \quad \text{if } n \equiv 3 \pmod 4,$$
$$\psi(h_1) = \psi(h_3) = \psi(h_2) = \psi(h_4) = \frac{3n+1}{4}e \quad \text{if } n \equiv 1 \pmod 4.$$

With the same reason, we can prove that

$$\psi(h_5) = \psi(h_6) = \psi(h_7) = \psi(h_8) = \frac{n+1}{4}e \quad \text{if } n \equiv 3 \pmod 4,$$
$$\psi(h_5) = \psi(h_6) = \psi(h_7) = \psi(h_8) = \frac{3n+1}{4}e \quad \text{if } n \equiv 1 \pmod 4.$$

Therefore $\text{supp}(\psi(S)) = \{\frac{n+1}{4}e\}$ if $n \equiv 3 \pmod 4$ and $\text{supp}(\psi(S)) = \{\frac{3n+1}{4}e\}$ if $n \equiv 1 \pmod 4$.

Suppose that $V = h_2h_3h_5h_7$. Since $\sigma(\phi(h_2h_3h_5h_7)) = \sigma(\phi(h_2h_3h_6h_8)) = 0$, we have that

$$\sigma(\psi(h_2h_3h_5h_7)) = \sigma(\psi(h_2h_3h_6h_8)) = e$$

which implies that $\sigma(\psi(h_5+h_7)) = \sigma(\psi(h_6+h_8))$. By $\psi(h_5+h_6) = \psi(h_7+h_8) = \frac{n+1}{2}e$ and $k_5 \le k_6$, $k_7 \le k_8$, we obtain that

$$\psi(h_5) = \psi(h_6) = \psi(h_7) = \psi(h_8) = \frac{n+1}{4}e \quad \text{if } n \equiv 3 \pmod 4,$$
$$\psi(h_5) = \psi(h_6) = \psi(h_7) = \psi(h_8) = \frac{3n+1}{4}e \quad \text{if } n \equiv 1 \pmod 4.$$

With the same reason, we can prove that

$$\psi(h_3) = \psi(h_4) = \psi(h_5) = \psi(h_6) = \frac{n+1}{4}e \quad \text{if } n \equiv 3 \pmod 4,$$
$$\psi(h_3) = \psi(h_4) = \psi(h_5) = \psi(h_6) = \frac{3n+1}{4}e \quad \text{if } n \equiv 1 \pmod 4.$$

Thus $\sigma(\psi(V)) = \sigma(\psi(h_2h_3h_5h_7)) = e$ implies that $\psi(h_2) = \psi(h_3) = \psi(h_5) = \psi(h_7)$ and hence $\psi(h_1+h_2) = \frac{n+1}{2}e$ implies that $\psi(h_1) = \psi(h_2)$. Therefore $\text{supp}(\psi(S)) = \{\frac{n+1}{4}e\}$ if $n \equiv 3 \pmod 4$ and $\text{supp}(\psi(S)) = \{\frac{3n+1}{4}e\}$ if $n \equiv 1 \pmod 4$.

Case 3. $|V| = 5$. Then $V = h_1h_2h_3h_5h_7$.

It follows that $\sigma(\phi(h_4h_6h_8)) = 0$ and hence $\sigma(\phi(h_4h_5h_7)) = \sigma(\phi(h_3h_6h_7)) = 0$. Thus by Property (*), we obtain $\sigma(\psi(h_4h_5h_7)) = \sigma(\psi(h_3h_6h_7)) = e$ which implies that $\sigma(\psi(h_4h_5)) = \sigma(\psi(h_3h_6)) = \frac{n+1}{2}e$ and $\psi(h_7) = \frac{n+1}{2}e$ by $\sigma(\psi(h_3h_4h_5h_6)) = e$. Therefore $\psi(h_8) = 0$, a contradiction to $k_7 \le k_8$. □

Lemma 4.4. *Let G, H, K and ϕ, ψ be as above. Let $K = \langle e \rangle$ and $S = h_1 \cdot \ldots \cdot h_8$ be a sequence over $G \setminus \{0\}$ with $\phi(h_1) = \phi(h_2) + \phi(h_3) = \phi(h_4) + \phi(h_5) = \phi(h_6) + \phi(h_7)$. If $\phi(S)$ is a squarefree sequence with $0 \notin \operatorname{supp}(\phi(S))$, then the following property $(*)$ does not hold.*

$$\begin{cases} \text{For any subsequence } V \text{ of } S \text{ with } \sigma(\phi(V)) = 0, \text{ we have that} \\ \sigma(\psi(V)) = \begin{cases} e, & \text{if } |V| = 3 \text{ or } 4, \\ e \text{ or } 2e, & \text{if } |V| = 5. \end{cases} \end{cases} \qquad (*)$$

Proof. Assume to the contrary that the property $(*)$ holds.

Since $\sigma(\phi(h_1h_2h_3)) = \sigma(\phi(h_2h_3h_4h_5)) = \sigma(\phi(h_4h_5h_1)) = \sigma(\phi(h_4h_5h_6h_7)) = \sigma(\phi(h_6h_7h_1)) = 0$, we obtain that $\sigma(\psi(h_1h_2h_3)) = \sigma(\psi(h_2h_3h_4h_5)) = \sigma(\psi(h_4h_5h_1)) = \sigma(\psi(h_4h_5h_6h_7))) = \sigma(\psi(h_6h_7h_1)) = e$ which implies that $\psi(h_1) = \psi(h_2) + \psi(h_3) = \psi(h_4) + \psi(h_5) = \psi(h_6) + \psi(h_7) = \frac{n+1}{2}e$.

Let $\psi(h_i) = k_ie$ where $1 \le i \le 8$ and $0 \le k_i \le n-1$. Without loss of generality, we can assume that $k_2 \le k_3$, $k_4 \le k_5$, $k_6 \le k_7$. We consider the sequence $W = h_1h_2h_4h_6h_8$ (see Figure 4.2).

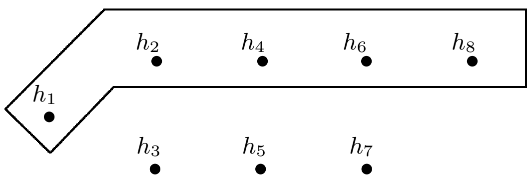

Figure 4.2:

Since $\phi(W) \in \mathscr{F}(\phi(G))$ and $\mathsf{D}(\phi(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $V \mid W$ such that $\sigma(\phi(V)) = 0$ and $|V| \in \{3, 4, 5\}$. We distinguish three cases depending on $|V|$.

Case 1. $|V| = 3$.

Obviously $h_1 \nmid v$. Then by symmetry, we only need to consider $V = h_2h_4h_6$ or $h = h_2h_4h_8$.

Suppose that $V = h_2h_4h_6$. Then $\sigma(\phi(h_2h_4h_6)) = \sigma(\phi(h_3h_5h_6)) = 0$ and hence $\sigma(\psi(h_2h_4h_6)) = \sigma(\psi(h_3h_5h_6)) = e$. Therefore $\sigma(\psi(h_2h_4)) = \sigma(\psi(h_3h_5)) = \frac{n+1}{2}e$ by $\sigma(\psi(h_2h_3h_4h_5)) = e$. It follows by that $\psi(h_6) = \frac{n+1}{2}e$, a contradiction to $k_6 \le k_7$ and $\psi(h_6) + \psi(h_7) = \frac{n+1}{2}e$.

Suppose that $V = h_2h_4h_8$. Then $\sigma(\phi(h_2h_4h_8)) = \sigma(\phi(h_3h_5h_8)) = 0$ and hence $\sigma(\psi(h_2h_4h_8)) = \sigma(\psi(h_3h_5h_8)) = e$. Therefore $\sigma(\psi(h_2h_4)) = \sigma(\psi(h_3h_5)) = \frac{n+1}{2}e$ by $\sigma(\psi(h_2h_3)) = \sigma(\psi(h_4h_5)) = \frac{n+1}{2}e$. It follows by $k_2 \le k_3$ and $k_4 \le k_5$ that $\psi(h_2) = \psi(h_3) = \psi(h_4) = \psi(h_5)$ and $\psi(h_8) = \frac{n+1}{2}e$. Therefore by $\sigma(\phi(h_1h_3h_4h_8)) = \sigma(\phi(h_2h_4h_8)) = 0$, we obtain that $\sigma(\psi(h_1h_3h_4h_8)) = e$. But $\sigma(\psi(h_1h_3h_4h_8)) = \frac{n+1}{2}e + \frac{n+1}{2}e + \frac{n+1}{2}e \ne e$, a contradiction.

Case 2. $|V| = 4$.

By symmetry, we only need to consider $V = h_1h_2h_4h_6$ or $V = h_2h_4h_6h_8$ or $V = h_1h_2h_4h_8$.

Suppose that $V = h_1h_2h_4h_6$. Then $\sigma(\phi(h_1h_2h_4h_6)) = \sigma(\phi(h_3h_4h_6)) = \sigma(\phi(h_2h_4h_7)) = 0$. Thus we obtain that $\sigma(\psi(h_3h_4h_6)) = \sigma(\psi(h_2h_4h_7)) = e$ which implies that $\sigma(\psi(h_3h_6)) = \sigma(\psi(h_2h_7)) = \frac{n+1}{2}e$ by $\sigma(\psi(h_2h_3h_6h_7)) = e$. Therefore $\psi(h_4) = \frac{n+1}{2}e$, a contradiction to $k_4 \le k_5$ and $\psi(h_4 + h_5) = \frac{n+1}{2}e$.

Suppose that $V = h_2h_4h_6h_8$. Then $\sigma(\phi(h_2h_4h_6h_8)) = \sigma(\phi(h_3h_5h_6h_8)) = 0$ and hence $\sigma(\psi(h_2h_4h_6h_8)) = \sigma(\psi(h_3h_5h_6h_8)) = e$. Thus $\sigma(\psi(h_2h_4)) = \sigma(\psi(h_3h_5)) = \sigma(\psi(h_6h_8)) = \frac{n+1}{2}e$. By $k_2 \le k_3, k_4 \le k_5$, we obtain that $\psi(h_2) = \psi(h_3) = \psi(h_4) = \psi(h_5)$. Since $\sigma(\phi(h_1h_3h_4h_6h_8)) = \sigma(\phi(h_2h_4h_6h_8)) = 0$, we obtain that $\sigma(\psi(h_1h_3h_4h_6h_8)) = \psi(h_1) + e \in \{e, 2e\}$. Thus $\psi(h_1) \in \{0, e\}$, a contradiction to $\psi(h_1) = \frac{n+1}{2}e$.

Suppose that $V = h_1h_2h_4h_8$. Then $\sigma(\phi(h_1h_2h_4h_8)) = \sigma(\phi(h_1h_3h_5h_8)) = 0$ and hence $\sigma(\psi(h_1h_2h_4h_8)) = \sigma(\psi(h_1h_3h_5h_8)) = e$. Thus $\sigma(\psi(h_2h_4)) = \sigma(\psi(h_3h_5)) = \sigma(\psi(h_1h_8)) = \frac{n+1}{2}e$. By $k_2 \le k_3, k_4 \le k_5$, we obtain that $\psi(h_2) = \psi(h_3) = \psi(h_4) = \psi(h_5)$. Since $\psi(h_1) = \frac{n+1}{2}e$, we obtain that $\psi(h_8) = 0$. It follows that $\sigma(\phi(h_3h_4h_8)) = \sigma(\phi(h_1h_2h_4h_8)) = 0$ and $\sigma(\psi(h_3h_4h_8)) = \frac{n+1}{2}e + 0 \neq e$, a contradiction.

Case 3. $|V| = 5$. Then $V = h_1h_2h_4h_6h_8$.

Then $\sigma(\phi(h_1h_2h_4h_6h_8)) = \sigma(\phi(h_2h_4h_7h_8)) = \sigma(\phi(h_3h_5h_7h_8)) = 0$ and hence $\sigma(\psi(h_2h_4h_7h_8)) = \sigma(\psi(h_3h_5h_7h_8)) = e$. Thus $\sigma(\phi(h_2h_4)) = \sigma(\phi(h_3h_5)) = \frac{n+1}{2}e$. By $k_2 \le k_3, k_4 \le k_5$, we obtain that $\psi(h_2) = \psi(h_3) = \psi(h_4) = \psi(h_5)$. Since $\sigma(\phi(h_1h_2h_4h_6h_8)) = \sigma(\phi(h_3h_4h_6h_8)) = 0$, we obtain that $\sigma(\psi(h_1h_2h_4h_6h_8)) = \psi(h_1) + \sigma(\psi(h_3h_4h_6h_8)) = \frac{n+1}{2}e + e \notin \{e, 2e\}$, a contradiction.

□

Lemma 4.5. *Let G, H, K and ϕ, ψ be as above. Let $K = \langle e \rangle$ and $S = h_1 \cdot \ldots \cdot h_8$ be a sequence over $G \setminus \{0\}$ with $\phi(h_1) = \phi(h_2) + \phi(h_3) = \phi(h_4) + \phi(h_5)$ and $\psi(h_3) = \psi(h_5) = \frac{n+1}{2}e$. If $\phi(S)$ is a squarefree sequence with $0 \notin \operatorname{supp}(\phi(S))$, then the following property $(*)$ does not hold.*

$$\begin{cases} \text{For any subsequence } V \text{ of } S \text{ with } \sigma(\phi(V)) = 0, \text{ we have that} \\ \sigma(\psi(V)) = \begin{cases} e, & \text{if } |V| = 3 \text{ or } 4, \\ e \text{ or } 2e, & \text{if } |V| = 5. \end{cases} \end{cases} \qquad (*)$$

Proof. Assume to the contrary that the property $(*)$ holds.

Since $\sigma(\phi(h_1)) = \sigma(\phi(h_2h_3)) = \sigma(\phi(h_4h_5))$, we obtain that $\sigma(\phi(h_1h_2h_3)) = \sigma(\phi(h_2h_3h_4h_5)) = \sigma(\phi(h_4h_5h_1)) = 0$ which implies that $\sigma(\psi(h_1h_2h_3)) = \sigma(\psi(h_2h_3h_4h_5)) = \sigma(\psi(h_4h_5h_1)) = e$. Therefore $\psi(h_1) = \frac{n+1}{2}e$ and $\psi(h_2) = \psi(h_4) = 0$ by $\psi(h_3) = \psi(h_5) = \frac{n+1}{2}e$.

We distinguish the following two cases to get contradictions to our assumption.

Case 1. $\psi(h_6) = \psi(h_7) = \psi(h_8) = \frac{n+1}{4}e$ if $n \equiv 3 \pmod 4$ and $\psi(h_6) = \psi(h_7) = \psi(h_8) = \frac{3n+1}{4}e$ if $n \equiv 1 \pmod 4$.

Consider the sequence $W = h_2h_4h_6h_7h_8$ (see Figure 4.3).

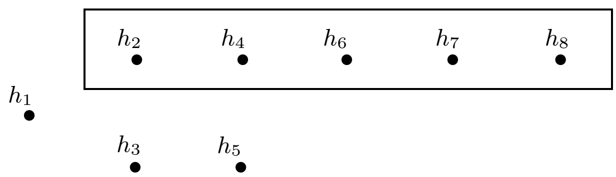

Figure 4.3:

Suppose that $n \equiv 3 \pmod 4$. Then $\psi(W) = 0^2(\frac{n+1}{4}e)^3$. Since $\phi(W) \in \mathscr{F}(\phi(G))$ and $\mathsf{D}(\phi(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $V \mid W$ such that $\sigma(\phi(V)) = 0$ and $|V| \in \{3, 4, 5\}$. If $|V| = 5$, then $\sigma(\psi(V)) = 3\frac{n+1}{4}e \notin \{e, 2e\}$, a contradiction. If $|V| = 4$, then $\sigma(\psi(V)) = e \in \{2\frac{n+1}{4}e, 3\frac{n+1}{4}e\}$, a contradiction. Thus $|V| = 3$ and $\sigma(\psi(V)) = e \in \{\frac{n+1}{4}e, 2\frac{n+1}{4}e, 3\frac{n+1}{4}e\}$ which implies that $n = 3$ and $h_2h_4 \,|\, V$. But $\sigma(\phi(Vh_3h_5(h_2h_4)^{-1})) = 0$ and $\sigma(\psi(Vh_3h_5(h_2h_4)^{-1}))) = 2\frac{n+1}{2}e + \frac{n+1}{4}e = 2e \neq e$, a contradiction.

Suppose that $n \equiv 1 \pmod 4$. Then $\psi(W) = 0^2(\frac{3n+1}{4}e)^3$. Since $\phi(W) \in \mathscr{F}(\phi(G))$ and $\mathsf{D}(\phi(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $V \mid W$ such that $\sigma(\phi(V)) = 0$ and $|V| \in \{3, 4, 5\}$. If $|V| = 5$, then $\sigma(\psi(V)) = 3\frac{3n+1}{4}e \notin \{e, 2e\}$ which implies that $n = 5$. Since $\sigma(\phi(h_3h_5h_6h_7h_8)) = 0$, we obtain that $\sigma(\psi(h_3h_5h_6h_7h_8)) = 3e + 3e + 4e + 4e + 4e = 3e \notin \{e, 2e\}$, a contradiction. If $|V| = 4$, then $\sigma(\psi(V)) = e \in \{2\frac{3n+1}{4}e, 3\frac{3n+1}{4}e\}$, a contradiction. Thus $|V| = 3$ and $\sigma(\psi(V)) = e \in \{\frac{3n+1}{4}e, 2\frac{3n+1}{4}e, 3\frac{3n+1}{4}e\}$, a contradiction.

Case 2. There exist distinct $i, j \in [6, 8]$ such that $\psi(h_i) + \psi(h_j) \neq \frac{n+1}{2}e$, say $\psi(h_6) + \psi(h_7) \neq \frac{n+1}{2}e$.

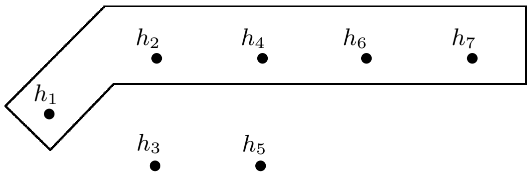

Figure 4.4:

Consider the sequence $W = h_1h_2h_4h_6h_7$ (see Figure 4.4).

Since $\phi(W) \in \mathscr{F}(\phi(G))$ and $\mathsf{D}(\phi(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $V \mid W$ such that $\sigma(\phi(V)) = 0$ and $|V| \in \{3, 4, 5\}$. We distinguish three cases depending on $|V|$.

Suppose that $|V| = 5$. Then $\sigma(\psi(V)) \neq \frac{n+1}{2}e + 0 + 0 + \frac{n+1}{2}e = e$ which implies that $\sigma(\psi(V)) = 2e$. Thus $\sigma(\psi(h_6h_7)) = \frac{n+3}{2}e$. It follows that $\sigma(\psi(h_1h_3h_5h_6h_7)) = 3e \notin \{e, 2e\}$, a contradiction to $\sigma(\phi(h_1h_3h_5h_6h_7)) = \sigma(\phi(V)) = 0$.

Suppose that $|V| = 4$. If $h_2h_4 \,|\, V$, then $\sigma(\phi(V(h_2h_4)^{-1}h_3h_5)) = 0$ and $\sigma(\psi(V(h_2h_4)^{-1}h_3h_5)) = 2e$, a contradiction. Thus $h_2h_4 \nmid V$. By symmetry, we only need to consider $V = h_1h_2h_6h_7$. But $\sigma(\psi(V)) \neq \frac{n+1}{2}e + 0 + \frac{n+1}{2}e = e$, a contradiction.

Suppose that $|V| = 3$. By symmetry, we only need to consider $V = h_1h_6h_7$, $V = h_2h_6h_7$, $V = h_2h_4h_6$ or $V = h_1h_2h_4$. If $V = h_1h_6h_7$, then $\sigma(\psi(V)) \neq e$, a contradiction. If $V = h_2h_6h_7$, then $\sigma(\phi(h_1h_3h_6h_7)) = \sigma(\phi(V)) = 0$ and hence $\sigma(\psi(h_1h_3h_6h_7)) = e$. It follows that $\psi(h_6) + \psi(h_7) = 0$ which implies that $\sigma(\psi(V)) = 0$, a contradiction. If $V = h_2h_4h_6$, then $\sigma(\psi(V)) = e$ which implies that $\psi(h_6) = e$. Thus $\sigma(\psi(h_3h_5h_6)) = 2e$, a contradiction to $\sigma(\phi(h_3h_5h_6)) = \sigma(\phi(V)) = 0$. If $V = h_1h_2h_4$, then $\phi(h_3) = \phi(h_4)$, a contradiction. □

Lemma 4.6. *Let G, H, K and ϕ, ψ be as above. Let $K = \langle e \rangle$ and $S = h_1 \cdot \ldots \cdot h_8$ be a sequence over $G \setminus \{0\}$ with $\phi(h_1) = \phi(h_2) + \phi(h_3) = \phi(h_4) + \phi(h_5)$ and $\psi(h_6) = \psi(h_7) = \frac{n+1}{2}e$. If $\phi(S)$ is a squarefree sequence with $0 \notin \operatorname{supp}(\phi(S))$, then the following property $(*)$ does not hold.*

$$\begin{cases} \text{For any subsequence } V \text{ of } S \text{ with } \sigma(\phi(V)) = 0, \text{ we have that} \\ \sigma(\psi(V)) = \begin{cases} e, & \text{if } |V| = 3 \text{ or } 4, \\ e \text{ or } 2e, & \text{if } |V| = 5. \end{cases} \end{cases} \qquad (*)$$

Proof. Assume to the contrary that the property $(*)$ holds.

Since $\phi(h_1) = \phi(h_2) + \phi(h_3) = \phi(h_4) + \phi(h_5)$, we obtain that $\sigma(\phi(h_1h_2h_3)) = \sigma(\phi(h_2h_3h_4h_5)) = \sigma(\phi(h_4h_5h_1)) = 0$ which implies that $\sigma(\psi(h_1h_2h_3)) = \sigma(\psi(h_2h_3h_4h_5)) = \sigma(\psi(h_4h_5h_1)) = e$. Therefore $\psi(h_1) = \psi(h_2) + \psi(h_3) = \psi(h_4) + \psi(h_5) = \frac{n+1}{2}e$.

Let $\psi(h_i) = k_ie$ where $1 \leq i \leq 8$ and $0 \leq k_i \leq n-1$. Without loss of generality, we can assume that $k_2 \leq k_3$, $k_4 \leq k_5$. Consider the sequence $W = h_1h_2h_4h_6h_7$ (see Figure 4.5).

Since $\phi(W) \in \mathscr{F}(\phi(G))$ and $\mathsf{D}(\phi(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $V \mid W$ such that $\sigma(\phi(V)) = 0$ and $|V| \in \{3, 4, 5\}$. We distinguish three cases depending on $|V|$.

Case 1. $|V| = 3$.

By symmetry, we only need to consider $V = h_1h_6h_7$, $V = h_2h_4h_6$ or $V = h_4h_6h_7$.

Suppose that $V = h_1h_6h_7$. Then $\sigma(\psi(V)) = \frac{n+3}{2}e \neq e$, a contradiction.

Suppose that $V = h_2h_4h_6$. Then $\sigma(\phi(h_3h_5h_6)) = \sigma(\phi(V)) = 0$ and hence $\sigma(\psi(h_3h_5h_6)) = \sigma(\psi(V)) = e$. Thus $\sigma(\psi(h_3h_5)) = \sigma(\psi(h_2h_4)) = \frac{n+1}{2}e$ which implies that $\psi(h_3) = \psi(h_4) = \psi(h_5) = \psi(h_6)$ by $k_2 \leq k_3, k_4 \leq k_5$. Therefore $\sigma(\psi(h_1h_3h_4h_6)) = 3\frac{n+1}{2} \neq e$, a contradiction to $\sigma(\phi(h_1h_3h_4h_6)) = \sigma(\phi(V)) = 0$.

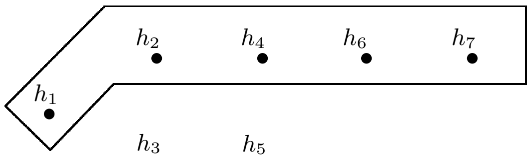

Figure 4.5:

Suppose that $V = h_4h_6h_7$. Then $\sigma(\phi(h_1h_5h_6h_7)) = \sigma(\phi(V)) = 0$ and hence $\sigma(\psi(h_1h_5h_6h_7)) = \sigma(\psi(h_4h_6h_7)) = e$. Thus $\psi(h_5) = \frac{n-1}{2}e$ and $\psi(h_4) = 0$, a contradiction to $\psi(h_4) + \psi(h_5) = \frac{n+1}{2}e$.

Case 2. $|V| = 4$.

By symmetry, we only need to consider $V = h_2h_4h_6h_7$, $V = h_1h_4h_6h_7$ or $V = h_1h_2h_4h_6$.

Suppose that $V = h_2h_4h_6h_7$. Then $\sigma(\phi(h_3h_5h_6h_7)) = \sigma(\phi(V)) = 0$ and hence $\sigma(\psi(h_3h_5h_6h_7)) = \sigma(\psi(V)) = e$. Thus $\sigma(\psi(h_3h_5)) = \sigma(\psi(h_2h_4)) = 0$, a contradiction to $\sigma(\psi(h_2h_3h_4h_5)) = e$.

Suppose that $V = h_1h_4h_6h_7$. Then $\sigma(\phi(h_5h_6h_7)) = \sigma(\phi(V)) = 0$ and hence $\sigma(\psi(h_5h_6h_7)) = e$. Thus $\psi(h_5) = 0$, a contradiction to $k_5 \geq k_4$ and $\psi(h_4) + \psi(h_5) = \frac{n+1}{2}e$.

Suppose that $V = h_1h_2h_4h_6$. Then $\sigma(\phi(h_1h_3h_5h_6)) = \sigma(\phi(V)) = 0$ and hence $\sigma(\psi(h_1h_3h_5h_6)) = \sigma(\psi(V)) = e$. Thus $\sigma(\psi(h_3h_5)) = \sigma(\psi(h_2h_4)) = 0$, a contradiction to $\sigma(\psi(h_2h_3h_4h_5)) = e$.

Case 3. $|V| = 5$.

Then $\sigma(\phi(h_3h_4h_6h_7)) = \sigma(\phi(h_2h_5h_6h_7)) = \sigma(\phi(V)) = 0$ which implies that $\sigma(\psi(h_3h_4h_6h_7)) = \sigma(\psi(h_2h_5h_6h_7)) = e$. Thus $\sigma(\psi(h_3h_4)) = \sigma(\psi(h_2h_5)) = 0$ which implies that $\sigma(\psi(h_2h_3h_4h_5)) = 0$, a contradiction to $\sigma(\phi(h_2h_3h_4h_5)) = 0$. □

Proposition 4.7. *Let G, H, K and ϕ, ψ be as above. Let $K = \langle e \rangle$ and S be a sequence over $G \setminus \{0\}$. If $\phi(S)$ is a squarefree sequence of length $|\phi(S)| = 8$ with $0 \notin \text{supp}(\phi(S))$ and satisfies the following property $(*)$:*

$$\begin{cases} \text{For any subsequence } V \text{ of } S \text{ with } \sigma(\phi(V)) = 0, \text{ we have that} \\ \sigma(\psi(V)) = \begin{cases} e, & \text{if } |V| = 3 \text{ or } 4, \\ e \text{ or } 2e, & \text{if } |V| = 5. \end{cases} \end{cases} \qquad (*)$$

then $\frac{n+1}{2}e \notin \text{supp}(\psi(S))$.

Proof. For any $v \in \phi(G) \setminus \{0\} = H \setminus \{0\}$, we define

$$\mathsf{N}_v(\phi(S)) = |\{T \mid \phi(S) \ : \ |T| = 2 \text{ and } \sigma(T) = v\}| + \delta_v,$$

where

$$\delta_v = \begin{cases} 1, & \text{if } v \in \mathsf{supp}(\phi(S)); \\ 0, & \text{if } v \notin \mathsf{supp}(\phi(S)). \end{cases}$$

We distinguish the following four cases.

Case 1. There exists $v \in H \setminus \{0\}$ such that $\mathsf{N}_v(\phi(S)) = 4$ and $\delta_v = 0$.

Without loss of generality, we can assume that $v = \phi(h_1 + h_2) = \phi(h_3 + h_4) = \phi(h_5 + h_6) = \phi(h_7 + h_8)$ where $S = h_1 \cdot \ldots \cdot h_8$. By Lemma 4.3, we obtain that $\frac{n+1}{2}e \notin \text{supp}(\psi(S))$.

Case 2. There exists $v \in H \setminus \{0\}$ such that $\mathsf{N}_v(\phi(S)) = 4$ and $\delta_v = 1$.

Without loss of generality, we can assume that $v = \phi(h_1) = \phi(h_2 + h_3) = \phi(h_4 + h_5) = \phi(h_6 + h_7)$ where $S = h_1 \cdot \ldots \cdot h_8$. By Lemma 4.4, we obtain that $\frac{n+1}{2}e \notin \text{supp}(\psi(S))$.

Now, we can assume that, for each $v \in H\setminus\{0\}$, $\mathsf{N}_v(\phi(S)) \le 3$. Since $\sum_{v\in H\setminus\{0\}} \mathsf{N}_v(\phi(S)) = \frac{8\times 7}{2}+8 = 36$ and $|H\setminus\{0\}| = 15$, by simple calculation, we obtain that $|\{v \in H\setminus\{0\} \mid \mathsf{N}_v(\phi(S)) = 3\}| \ge 6$. We continue with further case distinctions.

Case 3. There exist three distinct $v_1, v_2, v_3 \in H\setminus\{0\}$ such that $\mathsf{N}_{v_1}(\phi(S)) = \mathsf{N}_{v_2}(\phi(S)) = \mathsf{N}_{v_3}(\phi(S)) = 3$ and $\delta_{v_1} = \delta_{v_2} = \delta_{v_3} = 1$.

Let $S = h_1\cdot\ldots\cdot h_8$. For each $i \in [1,3]$, we denote by $A_i = \{v_i\}\cup\mathrm{supp}(\phi(T_{i_1}T_{i_2}))$, where $|T_{i_1}| = |T_{i_2}| = 2$, $T_{i_1}T_{i_2} \mid S$, and $v_i = \sigma(\phi(T_{i_1})) = \sigma(\phi(T_{i_2}))$. Thus $|A_i| = 5$ for each $i \in [1,3]$. By symmetry, we can distinguish the following two cases.

Subcase 3.1. There exists $i \in [1,3]$, say $i = 1$, such that $v_2 \notin A_1$ and $v_3 \notin A_1$.

Then we can assume that $v_1 = \phi(h_1) = \phi(h_2+h_3) = \phi(h_4+h_5)$, $v_2 = \phi(h_6)$, and $v_3 = \phi(h_7)$. It follows that $\psi(h_1) = \psi(h_6) = \psi(h_7) = \frac{n+1}{2}e$, a contradiction to Lemma 4.6.

Subcase 3.2. For each $i \in [1,3]$, there exists $j \in [1,3]$ such that $j \ne i$ and $v_j \in A_i$.

For $i = 1$, we can assume that $v_2 \in A_1$. Then $v_1 \in A_2$. For $i = 3$, we obtain that $v_1 \in A_3$ or $v_2 \in A_3$ which implies that $v_3 \in A_1$ or $v_3 \in A_2$. By symmetry, we can assume that $v_3 \in A_1$ and hence $v_2, v_3 \in A_1$.

Without loss of generality, we can assume that $v_1 = \phi(h_1) = \phi(h_2+h_3) = \phi(h_4+h_5)$, $v_2 = \phi(h_3)$, and $v_3 = \phi(h_5)$. It follows that $\psi(h_1) = \psi(h_3) = \psi(h_5) = \frac{n+1}{2}e$, a contradiction to Lemma 4.5.

Case 4. There exist three distinct $v_1, v_2, v_3 \in H\setminus\{0\}$ such that $\mathsf{N}_{v_1}(\phi(S)) = \mathsf{N}_{v_2}(\phi(S)) = \mathsf{N}_{v_3}(\phi(S)) = 3$ and $\delta_{v_1} = \delta_{v_2} = \delta_{v_3} = 0$.

Let $S = h_1\cdot\ldots\cdot h_8$. For each $i \in [1,3]$, we denote by $A_i = \mathrm{supp}(\phi(R_{i_1}R_{i_2}R_{i_3}))$, where $|R_{i_1}| = |R_{i_2}| = |R_{i_3}| = 2$, $R_{i_1}R_{i_2}R_{i_3} \mid S$, and $v_i = \sigma(\phi(R_{i_1})) = \sigma(\phi(R_{i_2})) = \sigma(\phi(R_{i_3}))$. Thus $|A_i| = 6$ for each $i \in [1,3]$ and hence $|A_i \cap A_j| \ge 6+6-8 = 4$ for distinct i, j where $1 \le i, j \le 3$. We proceed by the following two claims

Claim A. For each $i \in [1,3]$ and each $k \in [1,3]$, $\sigma(\psi(R_{i_k})) = \frac{n+1}{2}e$.

Proof of **Claim A**. For each $i \in [1,3]$, $\sigma(\phi(R_{i_1}R_{i_2})) = \sigma(\phi(R_{i_1}R_{i_3})) = \sigma(\phi(R_{i_2}R_{i_3})) = 0$ implies that $\sigma(\psi(R_{i_1}R_{i_2})) = \sigma(\psi(R_{i_1}R_{i_3})) = \sigma(\psi(R_{i_2}R_{i_3})) = e$. Thus $\sigma(\psi(R_{i_k})) = \frac{n+1}{2}e$ for all $k \in [1,3]$.

□(Proof of Claim A)

Claim B. For each $j \in [2,3]$, there exist $1 \le s < t \le 3$ such that $\mathrm{supp}(\phi(R_{1_s}R_{1_t})) \subseteq A_j$. Furthermore, there exist distinct $1 \le x, y \le 3$ such that $R_{1_s}R_{1_t} = R_{j_x}R_{j_y}$.

Proof of **Claim B**. Without loss of generality, we can assume that $j = 2$. Let $R_{1_1} = g_1g_2$, $R_{1_2} = g_3g_4$, and $R_{1_3} = g_5g_6$.

Since $|A_1 \cap A_2| \ge 4$, by symmetry, we only need to consider two cases: $\mathrm{supp}(\phi(g_1g_2g_3g_4)) \subseteq A_1\cap A_2$ and $\mathrm{supp}(\phi(g_1g_2g_3g_5)) = A_1 \cap A_2$.

Suppose that $\mathrm{supp}(\phi(g_1g_2g_3g_5)) = A_1 \cap A_2$. Then there exists $x \in [1,3]$ such that $R_{2_x} \mid g_1g_2g_3g_5$. By symmetry, there are only three cases: $R_{2_x} = g_1g_2$, $R_{2_x} = g_1g_3$, and $R_{2_x} = g_3g_5$. If $R_{2_x} = g_1g_2$, then $v_1 = v_2$, a contradiction. If $R_{2_x} = g_1g_3$, then $v_2 = \sigma(\phi(R_{2_x})) = \sigma(\phi(g_2g_4))$ which implies that $\phi(g_4) \in A_1 \cap A_2$, a contradiction. If $R_{2_x} = g_3g_5$, then $v_2 = \sigma(\phi(R_{2_x})) = \sigma(\phi(g_4g_6))$ which implies that $\mathrm{supp}(\phi(g_4g_6)) \subseteq A_1 \cap A_2$, a contradiction.

Suppose that $\mathrm{supp}(\phi(g_1g_2g_3g_4)) \subseteq A_1 \cap A_2$. Then $\mathrm{supp}(\phi(R_{1_1}R_{1_2})) \subseteq A_2$. Furthermore, there must exist $x \in [1,3]$ such that $R_{2_x} \mid R_{1_1}R_{1_2}$. Thus $\sigma(\phi(R_{2_x})) = \sigma(\phi(R_{1_1}R_{1_2}R_{2_x}^{-1}))$ which implies that $R_{1_1}R_{1_2}R_{2_x}^{-1} = R_{2_y}$ for some $y \in [1,3]\setminus\{x\}$. □(Proof of Claim B)

Without loss of generality, we can assume that $S = h_1 \cdot \ldots \cdot h_8$, $v_1 = \phi(h_1) + \phi(h_2) = \phi(h_3) + \phi(h_4) = \phi(h_5) + \phi(h_6)$.

If $|A_1 \cap A_2| = |A_1 \cap A_3| = 4$, then $v_2 = \sigma(\phi(h_7h_8)) = v_3$, a contradiction. Thus by symmetry and Claim B, we can assume that $|A_1 \cap A_2| = 5$ and $v_2 = \phi(h_1) + \phi(h_3) = \phi(h_2) + \phi(h_4) = \phi(h_5) + \phi(h_7)$ which implies that $\phi(h_1) + \phi(h_4) = \phi(h_2) + \phi(h_3) = \phi(h_6) + \phi(h_7)$ (See Figure 4.6).

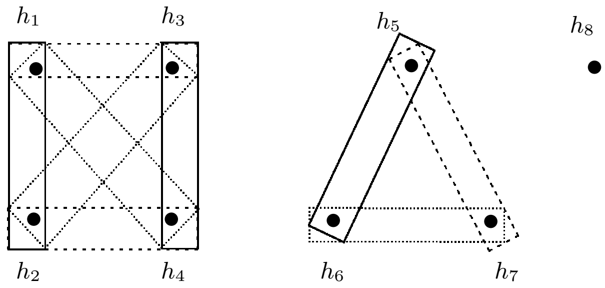

Figure 4.6:

Then we have

$$\psi(h_1 \cdot \ldots \cdot h_7) = (\frac{n+1}{4}e)^7 \quad \text{if } n \equiv 3 \pmod 4,$$
$$\psi(h_1 \cdot \ldots \cdot h_7) = (\frac{3n+1}{4}e)^7 \quad \text{if } n \equiv 1 \pmod 4.$$

Assume to the contrary that $\frac{n+1}{2}e \in \mathsf{supp}(\psi(S))$. Then $\psi(h_8) = \frac{n+1}{2}e$.

Consider the sequence $W = h_1h_2h_5h_7h_8$. Since $\phi(W) \in \mathscr{F}(\phi(G))$ and $\mathsf{D}(\phi(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $V \mid W$ such that $\sigma(\phi(V)) = 0$ and $|V| \in \{3, 4, 5\}$. If $|V| = 4$, by $\sigma(\psi(V)) = e$ we obtain that $V = h_1h_2h_5h_7$ which implies that $\phi(h_6) = \phi(h_7)$, a contradiction. If $|V| = 5$, then $\sigma(\psi(V)) = \frac{n+3}{2}e \notin \{e, 2e\}$, a contradiction.

Therefore $|V| = 3$. By $\sigma(\psi(V)) = 1$, we obtain that $h_8 \mid V$. Since $h_1h_2 \nmid V$ and $h_5h_7 \nmid V$, by symmetry, we only need to consider $V = h_1h_5h_8$. Then $\sigma(\phi(h_2h_3h_4h_5h_8)) = 0$ and $\sigma(\psi(h_2h_3h_4h_5h_8)) \notin \{e, 2e\}$, a contradiction.

Corollary 4.8. *Let G, H, K and ϕ, ψ be as above. Let $K = \langle e \rangle$ and S be a sequence over $G \setminus \{0\}$. If $\phi(S)$ is a squarefree sequence with $0 \notin \mathrm{supp}(\phi(S))$ and satisfies the following property $(*)$:*

$$\begin{cases} \text{For any subsequence } V \text{ of } S \text{ with } \sigma(\phi(V)) = 0, \text{ we have that} \\ \sigma(\psi(V)) = \begin{cases} e, & \text{if } |V| = 3 \text{ or } 4, \\ e \text{ or } 2e, & \text{if } |V| = 5. \end{cases} \end{cases} \qquad (*)$$

then $|S| \le 8$.

Proof. Assume to the contrary that $|S| \ge 9$. Without loss of generality, we can assume that $|S| = 9$.

If $\mathrm{supp}(\psi(S)) = \{0\}$, then $S \in \mathscr{F}(\ker(\psi))$. By $\mathsf{D}(\ker(\psi)) = \mathsf{D}(C_2^4) = 5$, S has a subsequence V of length $|V| \in [3, 5]$ such that $\sigma(\phi(V)) = 0$ and $\sigma(\psi(V)) = 0$, a contradiction.

Thus we can choose $w \mid S$ such that $\psi(w) \ne 0$. By Lemma 2.4, there exist distinct $g_1, g_2 \in \mathrm{supp}(Sw^{-1})$ such that $\phi(w) = \phi(g_1) + \phi(g_2)$. If there exist another $g_3, g_4 \in \mathrm{supp}(S(wg_1g_2)^{-1})$ such that $\phi(w) =$

$\phi(g_3)+\phi(g_4)$, then $\sigma(\phi(wg_1g_2))=\sigma(\phi(wg_3g_4))=\sigma(\phi(g_1g_2g_3g_4))=0$ which implies that $\sigma(\psi(wg_1g_2))=\sigma(\psi(wg_3g_4))=\sigma(\psi(g_1g_2g_3g_4))=e$. Therefore $\psi(w)=\frac{n+1}{2}e$, a contradiction to Proposition 4.7.

Otherwise, choose $g_3\in \operatorname{supp}(S(wg_1g_2)^{-1})$. Then $\phi(S(w+g_3)w^{-1})$ is a squarefree sequence of length 9. By Lemma 2.4, there exist distinct $g_i,g_j\in \operatorname{supp}(Sw^{-1})$ such that $\phi(w+g_3)=\phi(g_i)+\phi(g_j)$. Clearly, $g_ig_j\mid S(wg_1g_2g_3)^{-1}$ or $|\{g_1,g_2\}\cap\{g_i,g_j\}|=1$.

Suppose that $g_ig_j\mid S(wg_1g_2g_3)^{-1}$. Then $\sigma(\phi(wg_1g_2))=\sigma(\phi(wg_3g_ig_j))=\sigma(\phi(g_1g_2g_3g_ig_j))=0$ which implies that $\sigma(\psi(wg_1g_2))=\sigma(\psi(wg_3g_ig_j))=e$ and $\sigma(\psi(g_1g_2g_3g_ig_j))\in\{e,2e\}$. Therefore $\psi(w)\in\{\frac{n+1}{2}e,0\}$. It follows by the choice of w that $\psi(w)=\frac{n+1}{2}e$, a contradiction to Proposition 4.7.

Supppose that $|\{g_1,g_2\}\cap\{g_i,g_j\}|=1$. By symmetry, we can assume that $g_i=g_1$ and $g_j\in \operatorname{supp}(S(wg_1g_2g_3)^{-1})$. Then $\sigma(\phi(wg_1g_2))=\sigma(\phi(wg_1g_3g_j))=\sigma(\phi(g_1g_2g_1g_3g_j))=\sigma(\phi(g_2g_3g_j))=0$ which implies that $\sigma(\psi(wg_1g_2))=\sigma(\psi(wg_1g_3g_j))=\sigma(\psi(g_2g_3g_j))=e$. Therefore $\psi(g_2)=\frac{n+1}{2}e$, a contradiction to Proposition 4.7. □

5 The proof of Theorem 1.2.2

Proposition 5.1. *$\eta(C_2^3\oplus C_{2n})=2n+6$, where $n\ge 3$ is an odd integer.*

Proof. Let $G=H\oplus K$ be a finite abelian group, where $H\cong C_2^4$ and $K\cong C_n$ with $n\ge 3$ an odd integer. Denote ϕ to be the projection from G to H and ψ to be the projection from G to K.

In order to prove that $\eta(G)=2n+6$, by Lemma 2.3 we only need to prove that $\eta(G)\le 2n+6$. Assume to the contrary that there exists a sequence S of length $2n+6$ over G containing no short zero-sum subsequence.

Since $|\phi(S)|=|S|=2n+6=2(n-5)+16$, $\eta(C_2^4)=16$, and $\mathsf{D}(C_n)=n$, we obtain that S allows a product decomposition as

$$S=S_1\cdot\ldots\cdot S_r\cdot S_0,$$

where $S_1,\ldots,S_r,S_0$ are sequences over G and, for every $i\in[1,r]$, $\phi(S_i)$ has sum zero and length $|S_i|\le 2$. Therefore $\phi(S_0)$ is squarefree over $H\setminus\{0\}$ and $n-4\le r\le n-1$. We distinguish the following four cases depending on r to get contradictions.

Case 1. $r=n-1$. Then $|S_0|\ge 8$.

We proceed by the following assertion first

Assertion *A.* There exists an element $e\in\ker(\phi)=K$ such that $\sigma(S_1)\cdot\ldots\cdot\sigma(S_{n-1})=e^{n-1}$. Furthermore, for any element $h\mid SS_0^{-1}$, the sequence S_0h has the following property:

$$\begin{cases}\text{For any subsequence } V \text{ of } S_0h \text{ with } \sigma(\phi(V))=0, \text{ we have that}\\ \sigma(\psi(V))=\begin{cases} e, & \text{if } |V|=3 \text{ or } 4,\\ e \text{ or } 2e, & \text{if } |V|=5.\end{cases}\end{cases}$$

Proof of **Assertion A**. By our assumption and Lemma 2.1.1, $\sigma(S_1)\cdot\ldots\cdot\sigma(S_{n-1})$ is zero-sum free over K. Then there exists an element $e\in K\setminus\{0\}$ such that $\sigma(S_1)=\cdots=\sigma(S_{n-1})=e$.

Without loss of generality, we can assume that $h\mid S_{n-1}$. If $\sigma(\psi(V))=0$, then V is a short zero-sum subsequence of S, a contradiction. Thus $\sigma(\psi(V))\ne 0$.

If $\sigma(\psi(V))=ke$ with $k\in[2,n-1]$ and $|V|\in\{3,4\}$, then $S_1\cdot\ldots\cdot S_{n-k}\cdot V$ is a short zero-sum subsequence of S , a contradiction.

If $\sigma(\psi(V))=ke$ with $k\in[3,n-1]$ and $|V|=5$, then $S_1\cdot\ldots\cdot S_{n-k}\cdot V$ is a short zero-sum subsequence of S , a contradiction. □(Proof of Assertion A)

If $|S_0| > 8$, we obtain a contradiction to Corollary 4.8. Thus we can assume that $|S_0| = 8$ and hence $|S_i| = 2$ for each $i \in [1, n-1]$.

If $\mathrm{supp}(\phi(S)) \nsubseteq \mathrm{supp}(\phi(S_0))$, there exists $h \mid SS_0^{-1}$, such that $\phi(S_0 h)$ is squarefree. By Corollary 4.8, $|S_0 h| \le 8$, a contradiction. Thus $\mathrm{supp}(\phi(S)) \subseteq \mathrm{supp}(\phi(S_0))$. Without loss of generality, we can assume that $S_{n-1} = h_1 h_2$, $h_3 \mid S_0$ and $\phi(h_1) = \phi(h_2) = \phi(h_3)$. If there exist distinct $1 \le i, j \le 3$ such that $\psi(h_i + h_j) \ne e$, then $S_1 \cdot \ldots \cdot S_{n-2} \cdot h_i h_j$ has a short zero-sum subsequence, a contradiction. Therefore $\psi(h_1 + h_2) = \psi(h_1 + h_3) = \psi(h_2 + h_3) = e$ which implies that $\psi(h_1) = \psi(h_2) = \psi(h_3) = \frac{n+1}{2}e$ and hence $\frac{n+1}{2}e \in \mathrm{supp}(\phi(S_0))$, a contradiction to Proposition 4.7.

Case 2. $r = n-2$. Then $|S_0| \ge 10$.

By Lemma 4.1, S_0 has a subsequence T of length $|T| \in \{3, 4\}$ such that $\sigma(\phi(T)) = 0$ and $\sigma(\psi(T)) \ne \sigma(\psi(S_1))$. By our assumption, the sequences $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{n-2})\sigma(T)$ is zero-sum free over $\ker(\phi) = K$. By Lemma 2.1.1, we obtain that

$$\sigma(S_1) = \cdots = \sigma(S_{n-2}) = \sigma(T),$$

a contradiction.

Case 3. $r = n-3$. Then $|S_0| \ge 12$.

If $n = 3$, then by Lemma 4.2, S contains a short zero-sum subsequence, a contradiction. We can assume that $n \ge 5$.

By Lemma 2.4, there exist disjoint $T_1, T_2 \mid S_0$ such that $\sigma(\phi(T_1)) = \sigma(\phi(T_2)) = 0$ and $|T_1| = |T_2| = 3$. By our assumption, the sequence $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{n-3}) \cdot \sigma(T_1) \cdot \sigma(T_2)$ contains no zero-sum subsequence over $\ker(\phi) = K \cong C_n$, therefore by Lemma 2.1.1,

$$\sigma(S_1) = \cdots = \sigma(S_{n-3}) = \sigma(T_1) = \sigma(T_2) = e,$$

for some $e \in \ker(\phi) = K$ of order n.

Assertion B**.** Let V be a subsequence of S_0 with $\sigma(\phi(V)) = 0$. Then

$$\sigma(\psi(V)) = \begin{cases} e, & \text{if } |V| = 3, \\ e \text{ or } 2e, & \text{if } |V| = 4 \text{ or } 5. \end{cases}$$

Proof of **Assertion B**. If $|V| = 3$, then $|S_0 V^{-1}| = 12 - 3 = 9$. By Lemma 2.4, there exists $V_1 \mid S_0 V^{-1}$ such that $\sigma(\phi(V_1)) = 0$ and $|V_1| = 3$. By our assumption, the sequence $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{n-3}) \cdot \sigma(V) \cdot \sigma(V_1)$ contains no zero-sum subsequence over K. Therefore by Lemma 2.1.1,

$$\sigma(S_1) = \cdots = \sigma(S_{n-3}) = \sigma(V) = \sigma(V_1) = e.$$

If $|V| = 4$ or 5, by our assumption, $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{n-3})\sigma(V)$ is zero-sum free over K. Since $\sigma(S_1) = \cdots = \sigma(S_{n-3}) = e$, we obtain that $\Sigma(\sigma(S_1) \cdot \ldots \cdot \sigma(S_{n-3})) = \{e, \ldots, (n-3)e\}$. It follows that $\sigma(\psi(V)) \in \{e, 2e\}$. $\square$(Proof of Assertion B)

Suppose that $\mathsf{supp}(\psi(S_0)) \setminus \{0, \frac{n+1}{2}e\} \ne \emptyset$. Choose $u \mid S_0$ such that $\psi(u) \notin \{0, \frac{n+1}{2}e\}$. By Lemma 2.4, there exists a set $\{u_1, u_2, u_3, u_4\} \subseteq \mathsf{supp}(S_0 u^{-1})$ such that $\sigma(\phi(uu_1u_2)) = \sigma(\phi(uu_3u_4)) = \sigma(\phi(u_1u_2u_3u_4)) = 0$. Then by Assertion B, we deduce that $\sigma(\psi(uu_1u_2)) = \sigma(\psi(uu_3u_4)) = e$ and $\sigma(\psi(u_1u_2u_3u_4)) \in \{e, 2e\}$. Therefore $\psi(u_1 + u_2) = \psi(u_3 + u_4) \in \{e, \frac{n+1}{2}e\}$ and hence $\psi(u) \in \{0, \frac{n+1}{2}e\}$, a contradiction.

Suppose that $\mathsf{supp}(\psi(S_0)) \subseteq \{0, \frac{n+1}{2}e\}$. If there exists $v \mid S_0$ such that $\psi(v) = \frac{n+1}{2}e$, by Lemma 2.4, there exists a set $\{v_1, \ldots, v_8\} \subseteq \mathsf{supp}(S_0 v^{-1})$ such that

$$\sigma(\phi(vv_1v_2)) = \sigma(\phi(vv_3v_4)) = \sigma(\phi(vv_5v_6)) = \sigma(\phi(vv_7v_8)) = 0.$$

Thus

$$\sigma(\psi(vv_1v_2)) = \sigma(\psi(vv_3v_4)) = \sigma(\psi(vv_5v_6)) = \sigma(\psi(vv_7v_8)) = e$$

and

$$\psi(v_1 + v_2) = \psi(v_3 + v_4) = \psi(v_5 + v_6) = \psi(v_7 + v_8) = \frac{n+1}{2}e.$$

Since $\mathsf{supp}(\psi(S_0)) \subseteq \{0, \frac{n+1}{2}e\}$, we have $\psi(v_1 \cdot \ldots \cdot v_8) = 0^4(\frac{n+1}{2}e)^4$ which implies that $0^4 \mid \psi(S_0)$. Then we can always assume that $0^4 \mid \psi(S_0)$.

Choose $R \mid S_0$ such that $0^4 \mid \psi(R)$ and $|R| = 5$. By $\mathsf{D}(C_2^4) = 5$, there exists $R_1 \mid R$ such that $\sigma(\phi(R_1)) = 0$. By our assumption, $\sigma(\psi(R_1)) \neq 0$. It follows that $\sigma(\psi(R_1)) = \frac{n+1}{2}e \notin \{e, 2e\}$ by $n \geq 5$, a contradiction.

Case 4. $r = n - 4$. Then $|S_0| \geq 14$ and $n \geq 5$.

By Lemma 2.4, there exists a subsequence T_1 of S_0 such that $\sigma(\phi(T_1)) = 0$ and $|T_1| = 3$. Since $|S_0T_1^{-1}| = 11$, there exists a subsequence T_2 of $S_0T_1^{-1}$ such that $\sigma(\phi(T_2)) = 0$, $|T_2| \in \{3, 4\}$, and $\sigma(\psi(T_2)) \neq \sigma(\psi(T_1))$ by Lemma 4.1. By our assumption, the sequence $\sigma(S_1) \cdot \ldots \cdot \sigma(S_{n-4})\sigma(T_1)\sigma(T_2)$ contains no zero-sum subsequence. Therefore by Lemma 2.1.2, there exists an element $e \in K$ such that

$$\sigma(S_1) \cdot \ldots \cdot \sigma(S_{n-4}) \cdot \sigma(T_1) \cdot \sigma(T_2) = e^{n-3}(2e),$$

which implies that $\sigma(S_1) = \ldots = \sigma(S_{n-4}) = e$.

Again by Lemma 4.1, there exists a subsequence T_3 of S_0 such that $\sigma(\phi(T_3)) = 0$, $|T_3| \in \{3, 4\}$, and $\sigma(\psi(T_3)) \neq e$. Therefore $\sigma(\psi(T_3)) = 2e$ or $3e$.

Suppose that $\sigma(\psi(T_3)) = 2e$. Since $|S_0T_3^{-1}| \geq 10$, there exists a subsequence T_4 of $S_0T_3^{-1}$ such that $\sigma(\phi(T_4)) = 0$, $|T_4| \in \{3, 4\}$, and $\sigma(\psi(T_4)) = te$ with $t \in [2, n]$. If $t \geq 4$, then $S_1 \cdot \ldots \cdot S_{n-t} \cdot T_4$ is a short zero-sum subsequence of S, a contradiction. Otherwise $2 \leq t \leq 3$. Then $S_1 \cdot \ldots \cdot S_{n-t-2} \cdot T_3 \cdot T_4$ is a short zero-sum subsequence of S, a contradiction.

Suppose that $\sigma(\psi(T_3)) = 3e$. Since $|S_0T_3^{-1}| \geq 10$, there exists a subsequence T_4 of $S_0T_3^{-1}$ such that $\sigma(\phi(T_4)) = 0$, $|T_4| \in \{3, 4\}$, and $\sigma(\psi(T_4)) = te$ with $t \in [1, n] \setminus \{3\}$. If $t \geq 4$, then $S_1 \cdot \ldots \cdot S_{n-t} \cdot T_4$ is a short zero-sum subsequence of S, a contradiction. Otherwise $1 \leq t \leq 2$. Then $S_1 \cdot \ldots \cdot S_{n-3-t} \cdot T_3 \cdot T_4$ is a short zero-sum subsequence of S, a contradiction. □

Lemma 5.2. *Let (e_1, e_2, e_3, e) be a basis of $G = C_2^3 \oplus C_{2n}$ with $\mathrm{ord}(e_1) = \mathrm{ord}(e_2) = \mathrm{ord}(e_3) = 2$ and $\mathrm{ord}(e) = 2n$, where $n \geq 2$ is an even integer. Suppose that $\theta : G \to G$ is the homomorphism defined by $\theta(e_1) = e_1$, $\theta(e_2) = e_2$, $\theta(e_3) = e_3$, $\theta(e) = ne$ and $\zeta : G \to G$ is the homomorphism defined by $\zeta(e_1) = \zeta(e_2) = \zeta(e_3) = 0$, $\zeta(e) = e$.*

If S is a sequence of length $|S| = 8$ over G such that $\theta(S)$ is a squarefree sequence with $0 \notin \mathsf{supp}(\theta(S))$, then for any $k \in [1, n-1]$ and $\gcd(k, n) = 1$, there exists a subsequence T of S with length $|T| \in [3, 4]$ such that $\sigma(T) \in \ker(\theta)$ and $\sigma(T) \neq 2ke$.

Proof. Without loss of generality, we can assume that $k = 1$. Otherwise choose (e_1, e_2, e_3, ke) to be a basis of G.

Assume to the contrary that for all subsequences T of S with $|T| \in [3, 4]$ and $\sigma(T) \in \ker(\theta)$, we have that $\sigma(T) = 2e$.

For any $v \in \theta(G) \setminus \{0\}$, we define

$$\mathsf{N}_v(\theta(S)) = |\{T \mid \theta(S) \;:\; |T| = 2 \text{ and } \sigma(T) = v\}| + \delta_v,$$

where

$$\delta_v = \begin{cases} 1, & \text{if } v \in \mathsf{supp}(\theta(S)); \\ 0, & \text{if } v \notin \mathsf{supp}(\theta(S)). \end{cases}$$

Then $\sum_{v \in \theta(G) \setminus \{0\}} \mathsf{N}_v(\theta(S)) = \frac{8 \times 7}{2} + 8 = 36$ and $|\theta(G) \setminus \{0\}| = 15$ which implies that there exists an element $v \in \theta(G) \setminus \{0\}$ such that $\mathsf{N}_v(\theta(S)) \geq 3$. Therefore we can distinguish the following two cases.

Case 1. There exists $v \in \theta(G) \setminus \{0\}$ such that $\mathsf{N}_v(\theta(S)) \geq 3$ and $\delta_v = 1$.

Without loss of generality, we can assume that $\sigma(\theta(h_1)) = \sigma(\theta(h_2h_3)) = \sigma(\theta(h_4h_5))$. Then we have $\sigma(\theta(h_1h_2h_3)) = \sigma(\theta(h_2h_3h_4h_5)) = \sigma(\theta(h_4h_5h_1)) = 0$ which implies that $\sigma(\zeta(h_1h_2h_3)) = \sigma(\zeta(h_2h_3h_4h_5)) = \sigma(\zeta(h_4h_5h_1)) = 2e$. Therefore $\zeta(h_1) = \zeta(h_2 + h_3) = \zeta(h_4 + h_5) = e$ or $(n+1)e$.

Let $\zeta(h_i) = k_ie$, where $k_i \in [0, 2n-1]$ for each $i \in [1, 8]$. Then $k_2 + k_3, k_4 + k_5$ are odd and there exist distinct $i, j \in [6, 8]$ such that $k_i \equiv k_j \pmod 2$. Without loss of generality, we can assume that $k_2, k_4, k_6 + k_7$ are even and hence k_1, k_3, k_5 are odd. Consider the sequence $W = h_1h_2h_4h_6h_7$ (see Figure 5.7).

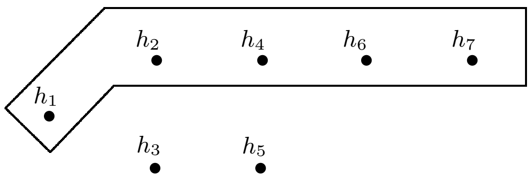

Figure 5.7:

Since $\theta(W) \in \mathscr{F}(\theta(G))$ and $\mathsf{D}(\theta(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $V \mid W$ such that $\sigma(\theta(V)) = 0$ and $|V| \in \{3, 4, 5\}$. We distinguish three cases depending on $|V|$.

Suppose that $|V| = 5$. Then $\sigma(\zeta(V)) = (2k+1)e$ for some $k \in [0, n-1]$, a contradiction to $\sigma(\theta(V)) = 0$.

Suppose that $|V| = 4$. If $h_2h_4 \mid V$, then $\sigma(\theta(V)) = \sigma(\theta(V(h_2h_4)^{-1}h_3h_5)) = \sigma(\theta(h_2h_3h_4h_5)) = 0$ and hence $\sigma(\zeta(V)) = \sigma(\zeta(V(h_2h_4)^{-1}h_3h_5)) = \sigma(\zeta(h_2h_3h_4h_5)) = 2e$. Therefore $\zeta(h_2 + h_4) = e$ or $(n+1)e$, a contradiction to k_2, k_4 are even. Thus, without loss of generality, we only need to consider $V = h_1h_2h_6h_7$. Then $\sigma(\zeta(V)) = (2k+1)e$ for some $k \in [0, n-1]$, a contradiction to $\sigma(\theta(V)) = 0$.

Suppose that $|V| = 3$. By symmetry, we only need to consider $V = h_1h_6h_7$, $V = h_2h_6h_7$, $V = h_2h_4h_6$ or $V = h_1h_2h_4$. If $V = h_1h_6h_7$, then $\sigma(\zeta(V)) = (2k+1)e$ for some $k \in [0, n-1]$, a contradiction to $\sigma(\theta(V)) = 0$. If $V = h_2h_6h_7$, then $\sigma(\theta(h_1h_3h_6h_7)) = \sigma(\theta(V)) = \sigma(\theta(h_1h_2h_3)) = 0$ and hence $\sigma(\zeta(h_1h_3h_6h_7)) = \sigma(\zeta(V)) = \sigma(\zeta(h_1h_2h_3)) = 2e$. It follows that $\zeta(h_2) = \zeta(h_1 + h_3) = \zeta(h_6 + h_7) = e$ or $(n+1)e$, a contradiction. If $V = h_2h_4h_6$, then $\sigma(\zeta(V)) = \sigma(\zeta(h_3h_5h_6)) = \sigma(\zeta(h_2h_3h_4h_5)) = 2e$ which implies that $\zeta(h_2 + h_4) = e$ or $(n+1)e$, a contradiction to k_2, k_4 are even. If $V = h_1h_2h_4$, then $\theta(h_3) = \theta(h_4)$, a contradiction.

Case 2. There exists $v \in \theta(G) \setminus \{0\}$ such that $\mathsf{N}_v(\theta(S)) \geq 3$ and $\delta_v = 0$.

Without loss of generality, we can assume that $\theta(h_1 + h_2) = \theta(h_3 + h_4) = \theta(h_5 + h_6)$. Then $\sigma(\theta(h_1h_2h_3h_4)) = \sigma(\theta(h_3h_4h_5h_6)) = \sigma(\theta(h_5h_6h_1h_2)) = 0$ and hence $\sigma(\zeta(h_1h_2h_3h_4)) = \sigma(\zeta(h_3h_4h_5h_6)) = \sigma(\zeta(h_5h_6h_1h_2)) = 2e$. Therefore $\zeta(h_1 + h_2) = \zeta(h_3 + h_4) = \zeta(h_5 + h_6) = e$ or $(n+1)e$.

Let $\zeta(h_i) = k_ie$, where $k_i \in [0, 2n-1]$ for each $i \in [1, 8]$. Without loss of generality, we can assume that k_2, k_4, k_6 are even and k_1, k_3, k_5 are odd.

Therefore we can distinguish the following two cases.

Subcase 2.1. k_7, k_8 are odd.

Consider the sequence $W = h_1h_3h_5h_7h_8$ (see Figure 5.8).

Since $\theta(W) \in \mathscr{F}(\theta(G))$ and $\mathsf{D}(\theta(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $V \mid W$ such that $\sigma(\theta(V)) = 0$ and $|V| \in \{3, 4, 5\}$. We distinguish two cases depending on $|V|$.

Suppose that $|V| = 5$ or 3. Then $\sigma(\zeta(V)) = (2k+1)e$ for some $k \in [0, n-1]$, a contradiction to $\sigma(\theta(V)) = 0$.

Suppose that $|V| = 4$. By symmetry, we only need to consider $V = h_1h_3h_5h_7$ or $V = h_1h_3h_7h_8$. For both cases, $h_1h_3 \mid V$. Since $\sigma(\theta(V)) = \sigma(\theta(V(h_1h_3)^{-1}h_2h_4)) = \sigma(\theta(h_1h_2h_3h_4)) = 0$, we obtain that

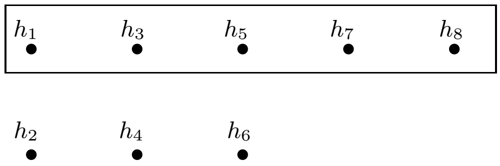

Figure 5.8:

$\sigma(\zeta(V)) = \sigma(\zeta(V(h_1h_3)^{-1}h_2h_4)) = \sigma(\zeta(h_1h_2h_3h_4)) = 2e$ which implies that $\zeta(h_1+h_3) = e$ or $(n+1)e$, a contradiction to k_1, k_3 are odd.

Subcase 2.2. k_7 or k_8 is even. Say, k_7 is even.

Consider the sequence $W = h_1h_2h_4h_6h_7$ (see Figure 5.9).

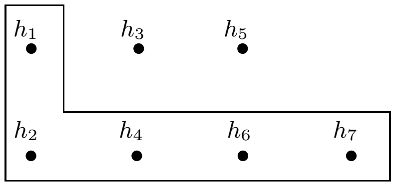

Figure 5.9:

Since $\theta(W) \in \mathscr{F}(\theta(G))$ and $\mathsf{D}(\theta(G)) = \mathsf{D}(C_2^4) = 5$, there exists a subsequence $V \mid W$ such that $\sigma(\theta(V)) = 0$ and $|V| \in \{3,4,5\}$. We distinguish three cases depending on $|V|$.

Suppose that $|V| = 5$. Then $\sigma(\zeta(V)) = (2k+1)e$ for some $k \in [0, n-1]$, a contradiction to $\sigma(\theta(V)) = 0$.

Suppose that $|V| = 4$. Since $\sigma(\theta(V)) = 0$, we obtain that $V = h_2h_4h_6h_7$. Then $\sigma(\theta(V)) = \sigma(\theta(h_1h_3h_6h_7)) = \sigma(\theta(h_1h_2h_3h_4)) = 0$ and hence $\sigma(\zeta(V)) = \sigma(\zeta(h_1h_3h_6h_7)) = \sigma(\zeta(h_1h_2h_3h_4)) = 2e$. Therefore $\zeta(h_1+h_3) = e$ or $(n+1)e$, a contradiction to k_1, k_3 are odd.

Suppose that $|V| = 3$. Since $\sigma(\theta(V)) = 0$, we obtain that $h_1 \nmid V$. By symmetry, we only need to consider $V = h_2h_4h_6$ or $V = h_2h_4h_7$. For both cases, $h_2h_4 \mid V$. Since $\sigma(\theta(V)) = \sigma(\theta(V(h_2h_4)^{-1}h_1h_3)) = \sigma(\theta(h_1h_2h_3h_4)) = 0$, we obtain that $\sigma(\zeta(V)) = \sigma(\zeta(V(h_2h_4)^{-1}h_1h_3)) = \sigma(\zeta(h_1h_2h_3h_4)) = 2e$ which implies that $\zeta(h_1+h_3) = e$ or $(n+1)e$, a contradiction to k_1, k_3 are odd. $\square$

Proposition 5.3. *$\eta(C_2^3 \oplus C_{2n}) = 2n+6$, where $n \geq 2$ is an even integer.*

Proof. Let $G = C_2^3 \oplus C_{2n}$, where $n \geq 2$ is an even integer. Suppose that $\theta : G \to G$ is the homomorphism defined by $\theta(e_1) = e_1$, $\theta(e_2) = e_2$, $\theta(e_3) = e_3$, $\theta(e) = ne$ and $\zeta : G \to G$ is the homomorphism defined by $\zeta(e_1) = \zeta(e_2) = \zeta(e_3) = 0$, $\zeta(e) = e$.

In order to prove that $\eta(G) = 2n+6$, by Lemma 2.3 we only need to prove that $\eta(G) \leq 2n+6$. Assume to the contrary that there exists a sequence S of length $2n+6$ over G containing no short zero-sum subsequence.

Since $|\theta(S)| = |S| = 2n+6 = 2(n-5)+16$, $\eta(C_2^4) = 16$, and $\mathsf{D}(C_n) = n$, we obtain that S allows a product decomposition as

$$S = S_1 \cdot \ldots \cdot S_r \cdot S_0,$$

where $S_1, \ldots, S_r, S_0$ are sequences over G and, for every $i \in [1, r]$, $\theta(S_i)$ has sum zero and length $|S_i| \leq 2$. What's more, $\theta(S_0)$ has no zero-sum subsequence of length ≤ 2 and $n-4 \leq r \leq n-1$. We distinguish the following four cases depending on r to get contradictions.

Case 1. $r = n-1$. Then $|S_0| \ge 8$.

Since S has no short zero-sum subsequence, we obtain that $\sigma(S_1)\cdot\ldots\cdot\sigma(S_{n-1})$ is zero-sum free over $\ker(\theta) \cong C_n$. By Lemma 2.1.1, there exists an element $k \in [1, n-1]$ such that $\sigma(S_1) = \cdots = \sigma(S_{n-1}) = 2ke$ and $\gcd(k, n) = 1$. Without loss of generality, we can assume that $k = 1$.

Since $\theta(S_0)$ has no zero-sum subsequence of length ≤ 2, by Lemma 5.2 we obtain that there exists a subsequence V of S_0 with length $|V| \in [3, 4]$ such that $\sigma(V) = 2te$ where $t \in [2, n]$. Then by calculation we get

$$\sigma(S_1 \cdot \ldots \cdot S_{n-t} \cdot V) = 0, \quad \text{and}$$
$$|S_1 \cdot \ldots \cdot S_{n-t} \cdot V| \le 2(n-t) + 4 \le 2n,$$

which implies that $S_1 \cdot \ldots \cdot S_{n-t} \cdot V$ is a short zero-sum subsequence of S, a contradiction.

Case 2. $r = n-2$. Then $|S_0| \ge 10$.

Since $\theta(S_0)$ is squarefree, by Lemma 2.4, S_0 has a subsequence T of length 3 such that $\sigma(\theta(T)) = 0$. By our assumption, the sequence $\sigma(S_1)\cdot\ldots\cdot\sigma(S_{n-2})\sigma(T)$ is zero-sum free over $\ker(\theta) \cong C_n$. By Lemma 2.1.1, we obtain that

$$\sigma(S_1) = \cdots = \sigma(S_{n-2}) = \sigma(T) = 2ke, \quad \text{for some } k \in [1, n-1] \text{ and } \gcd(k, n) = 1.$$

Without loss of generality, we can assume that $k = 1$. By Lemma 5.2, S_0 has a subsequence T' of length $|T'| \in \{3, 4\}$ such that $\sigma(\theta(T')) = 0$ and $\sigma(T') = 2te$ with $t \in [2, n]$. Therefore the sequence $S_1 \cdot \ldots \cdot S_{n-t} \cdot T'$ is a short zero-sum subsequence of S, a contradiction.

Case 3. $r = n-3$. Then $|S_0| \ge 12$ and $n \ge 4$.

By Lemma 2.4, there exist two subsequences T_1, T_2 of S_0 such that $\sigma(\theta(T_1)) = \sigma(\theta(T_2)) = 0$ and $|T_1| = |T_2| = 3$.

By our assumption, the sequence $\sigma(S_1)\cdot\ldots\cdot\sigma(S_{n-3})\cdot\sigma(T_1)\cdot\sigma(T_2)$ contains no zero-sum subsequence. Therefore by Lemma 2.1.1, there exists an element $k \in [1, n-1]$ and $\gcd(k, n) = 1$ such that

$$\sigma(S_1)\cdot\ldots\cdot\sigma(S_{n-3})\cdot\sigma(T_1)\cdot\sigma(T_2) = (2ke)^{n-1}.$$

Without loss of generality, we can assume that $k = 1$. By Lemma 5.2, there exists $T_1' \mid S_0$ such that $\sigma(\theta(T_1')) = 0$, $|T_1'| \in [3, 4]$, and $\sigma(T_1') = 2t_1e$ with $t_1 \in [2, n]$. By Lemma 5.2 again, there exists $T_2' \mid S_0(T_1')^{-1}$ such that $\sigma(\theta(T_2')) = 0$, $|T_2'| \in [3, 4]$, and $\sigma(T_2') = 2t_2e$ with $t_2 \in [2, n]$.

If $t_1 \ge 3$, then $S_1 \cdot \ldots \cdot S_{n-t_1} \cdot T_1'$ is a short zero-sum subsequence of S, a contradiction. If $t_2 \ge 3$, then $S_1 \cdot \ldots \cdot S_{n-t_2} \cdot T_2'$ is a short zero-sum subsequence of S, a contradiction. Otherwise $t_1 + t_2 = 4 \le n$. Then $S_1 \cdot \ldots \cdot S_{n-4} \cdot T_1' \cdot T_2'$ is a short zero-sum subsequence of S, a contradiction.

Case 4. $r = n-4$. Then $|S_0| \ge 14$ and $n \ge 4$.

We distinguish two cases depending on n.

Subcase 4.1. $n \ge 6$.

By Lemma 2.4, there exist two disjoint subsequences T_1, T_2 of S_0 such that $\sigma(\theta(T_1)) = \sigma(\theta(T_2)) = 0$ and $|T_1| = |T_2| = 3$.

By our assumption, the sequence $\sigma(S_1)\cdot\ldots\cdot\sigma(S_{n-4})\cdot\sigma(T_1)\cdot\sigma(T_2)$ contains no zero-sum subsequence. Therefore by Lemma 2.1.2, there exists an element $k \in [1, n-1]$ and $\gcd(k, n) = 1$ such that

$$\sigma(S_1)\cdot\ldots\cdot\sigma(S_{n-4})\cdot\sigma(T_1)\cdot\sigma(T_2) = (2ke)^{n-3}4ke \text{ or } (2ke)^{n-2}.$$

Without loss of generality, we can assume that $k = 1$ and $\sigma(T_1) = 2e$. By Lemma 5.2, there exists $T_3 \mid S_0(T_1)^{-1}$ such that $\sigma(\theta(T_3)) = 0$, $|T_3| \in [3, 4]$, and $\sigma(T_3) \ne 2e$. Then the sequence $\sigma(S_1)\cdot\ldots\cdot\sigma(S_{n-4})\cdot\sigma(T_1)\cdot\sigma(T_3)$ contains no zero-sum subsequence and hence $\sigma(S_1)\cdot\ldots\cdot\sigma(S_{n-4}) = (2e)^{n-4}$.

By Lemma 5.2, there exists $T_1' \mid S_0$ such that $\sigma(\theta(T_1')) = 0$, $|T_1'| \in [3,4]$, and $\sigma(T_1') = 2t_1e$ with $t_1 \in [2,n]$. By Lemma 5.2 again, there exists $T_2' \mid S_0(T_1')^{-1}$ such that $\sigma(\theta(T_2')) = 0$, $|T_2'| \in [3,4]$, and $\sigma(T_2') = 2t_2e$ with $t_2 \in [2,n]$.

If $t_1 \geq 4$, then $S_1 \cdot \ldots \cdot S_{n-t_1} \cdot T_1'$ is a short zero-sum subsequence of S, a contradiction. If $t_2 \geq 4$, then $S_1 \cdot \ldots \cdot S_{n-t_2} \cdot T_2'$ is a short zero-sum subsequence of S, a contradiction. Otherwise $t_1 + t_2 \leq 6 \leq n$. Then $S_1 \cdot \ldots \cdot S_{n-t_1-t_2} \cdot T_1' \cdot T_2'$ is a short zero-sum subsequence of S, a contradiction.

Subcase 4.2. $n = 4$. Then $S = S_0$.

By Lemma 2.4, there exist two disjoint subsequences T_1, T_2 of S_0 such that $\sigma(\theta(T_1)) = \sigma(\theta(T_2)) = 0$ and $|T_1| = |T_2| = 3$.

If $\sigma(T_1) \neq \sigma(T_2)$, since T_1T_2 can not be zero-sum, without loss of generality, we can assume that $\sigma(T_1) = 2e$ and $\sigma(T_2) = 4e$. By Lemma 5.2, there exists $T_3 \mid S_0(T_1T_2)^{-1}$ such that $\sigma(\theta(T_3)) = 0$, $|T_3| \in [3,4]$, and $\sigma(T_3) = 2te$ with $t \in [2,4]$. Thus, one of the sequences T_3, T_1T_3, T_2T_3 must be a short zero-sum subsequence of S, a contradiction.

Then $\sigma(T_1) = \sigma(T_2)$, since T_1T_2 can not be zero-sum, without loss of generality, we can assume that $\sigma(T_1) = \sigma(T_2) = 2e$.

We claim that for any subsequence T of S satisfying that $|T| = 3$ and $\sigma(\theta(T)) = 0$, we have $\sigma(T) = 2e$.

In fact, T_1 or T_2 must be disjoint with T. We can assume that T_1 and T are disjoint. If $\sigma(T) = 6e$, then T_1T is a short zero-sum subsequence, a contradiction. If $\sigma(T) = 4e$, we can do it as before to obtain a contradiction. Then $\sigma(T) = 2e$.

Since $\sigma(T_1) = 2e$, we can choose $g \mid T_1$ such that $\zeta(g) \notin \{e, 5e\}$. By Lemma 2.4, there exist subsequences $R_1, \ldots, R_4$ of ST_1^{-1} such that $\theta(g) = \sigma(\theta(R_1)) = \ldots = \sigma(\theta(R_4))$ and $|R_1| = \ldots = |R_4| = 2$. Since for each $i \in [1,6]$, $\sigma(\theta(gR_i)) = 0$, we obtain that $\sigma(gR_i) = 2e$. Thus $\sigma(\zeta(R_i)) = 2e - \zeta(g)$ for each $i \in [1,4]$. By $\sigma(\theta(R_1R_2)) = \sigma(\theta(R_1R_2)) = 0$, we have $\sigma(R_1R_2) = \sigma(R_1R_2) = 4e - 2\zeta(g)$. If $\sigma(R_1R_2) = 2e$, then $\zeta(g) \in \{e, 5e\}$, a contradiction. If $\sigma(R_1R_2) = 4e$, then $R_1R_2R_3R_4$ is a short zero-sum subsequence of S, a contradiction. Otherwise $\sigma(R_1R_2) = 6e$. Then $T_1R_1R_2$ is a short zero-sum subsequence of S, a contradiction. □

Proof of Theorem 1.2.2. By Proposition 5.1 and 5.3, it follows that $\eta(G) = 2n + 6$. If $n \geq 36 = \max\{2|C_2^3| + 1, 4|C_2^3| + 4\}$, by Lemma 2.2, we have that $\mathsf{s}(C_2^3 \oplus C_{2n}) = \eta(C_2^3 \oplus C_{2n}) + \exp(C_2^3 \oplus C_{2n}) - 1 = 2n + 6 + 2n - 1 = 4n + 5$. □

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References

[1] G. Bhowmik and J.-C. Schlage-Puchta, *Davenport's constant for groups of the form* $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{3d}$, Additive Combinatorics (A. Granville, M.B. Nathanson, and J. Solymosi, eds.), CRM Proceedings and Lecture Notes, vol. 43, American Mathematical Society, 2007, pp. 307 – 326.

[2] F. Chen and S. Savchev, *Long minimal zero-sum sequences in the groups* $C_2^{r-1} \oplus C_{2k}$, Integers **14** (2014), Paper A23.

[3] Y. Edel, C. Elsholtz, A. Geroldinger, S. Kubertin, and L. Rackham, *Zero-sum problems in finite abelian groups and affine caps*, The Quarterly Journal of Mathematics **58** (2007), no. 2, 159–186.

[4] P. Erdös, A. Ginzburg, and A. Ziv, *A theorem in additive number theory*, Bull. Res. Council Israel 10F (1961), 41-43.

[5] Y. Fan, W. Gao, and Q. Zhong, *On the Erdős-Gizburg-Ziv constant of finite abelian groups of high rank*, J. Number Theory, 131 (2011) 1864-1874.

[6] Y. Fan, W. Gao, L. Wang, and Q. Zhong, *Two zero-sum invariants on finite abelian groups*, European. J. Combin. 34(2013), 1331-1337.

[7] W. Gao, *On zero sum subsequences of restricted size II*, Discrete Math. **271** (2003), 51 – 59.

[8] W. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. 24 (2006) 337–369.

[9] W. Gao, A. Geroldinger, and W.A. Schmid, *Inverse zero-sum problems*, Acta Arith. **128** (2007), 245 – 279.

[10] W. Gao, Q.H. Hou, W.A. Schmid, and R. Thangadurai, *On short zero-sum subsequences II*, Integers **7** (2007), Paper A21, 22p.

[11] A. Geroldinger, *Additive group theory and non-unique factorizations*, Combinatorial Number Theory and Additive Group Theory (A. Geroldinger and I. Ruzsa, eds.), Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, 2009, pp. 1–86.

[12] A. Geroldinger and F. Halter-Koch, *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics, vol. 278, Chapman & Hall/CRC, 2006.

[13] A. Geroldinger, M. Liebmann, and A. Philipp, *On the Davenport constant and on the structure of extremal sequences*, Period. Math. Hung. **64** (2012), 213 – 225.

[14] A. Potechin, *Maximal caps in* $AG(6,3)$, Des. Codes Cryptography **46** (2008), 243 – 259.

[15] W.A. Schmid, *The inverse problem associated to the Davenport constant for* $C_2 \oplus C_2 \oplus C_{2n}$*, and applications to the arithmetical characterization of class groups*, Electron. J. Comb. **18(1)** (2011), Research Paper 33.

[16] W.A. Schmid and J.J. Zhuang, *On short zero-sum subsequences over p-groups*, Ars Comb. **95** (2010), 343 – 352.