ON ELASTICITIES OF LOCALLY FINITELY GENERATED MONOIDS

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Abstract. Let H be a commutative and cancellative monoid. The elasticity $\rho(a)$ of a non-unit $a \in H$ is the supremum of m/n over all m, n for which there are factorizations of the form $a = u_1 \cdot \ldots \cdot u_m = v_1 \cdot \ldots \cdot v_n$, where all u_i and v_j are irreducibles. The elasticity $\rho(H)$ of H is the supremum over all $\rho(a)$. We establish a characterization, valid for finitely generated monoids, when every rational number q with $1 < q < \rho(H)$ can be realized as the elasticity of some element $a \in H$. Furthermore, we derive results of a similar flavor for locally finitely generated monoids (they include all Krull domains and orders in Dedekind domains satisfying certain algebraic finiteness conditions) and for weakly Krull domains.

1. Introduction

In this paper, a *monoid* means a commutative cancellative semigroup with identity element and the monoids we mainly have in mind are multiplicative monoids of nonzero elements of domains. A monoid is said to be locally finitely generated if for every given element a there are only finitely many irreducibles (up to associates) which divide some power of a. Krull domains are locally finitely generated and more examples are given in Section 2 (see Examples 2.1).

Let H be a monoid and $a \in H$. If a has a factorization into irreducibles, say $a = u_1 \cdot \ldots \cdot u_k$, then k is called a factorization length and the set $\mathsf{L}(a) \subset \mathbb{N}$ of all possible factorization lengths is called the set of lengths of a. For convenience we set $\mathsf{L}(a) = \{0\}$ if a is a unit. The monoid H is said to be a BF-monoid if every non-unit has a factorization into irreducibles and all sets of lengths are finite. It is well-known that v-noetherian monoids are BF-monoids.

Suppose that H is a BF-monoid. The system $\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$ of sets of lengths, and all parameters controlling $\mathcal{L}(H)$, are a well-studied means to describe the non-uniqueness of factorizations of BF-monoids. Besides distances of factorizations, elasticities belong to the main parameters. For a finite set $L \subset \mathbb{N}$, $\rho(L) = \max L / \min L$ denotes the elasticity of L and, for an element $a \in H$, the elasticity $\rho(a)$ of a is the elasticity of its set of lengths. The elasticity $\rho(H)$ of H is the supremum of $\rho(L)$ over all $L \in \mathcal{L}(H)$.

Since the late 1980s various aspects of elasticities have found wide attention in the literature. We refer to a survey by David F. Anderson [6] for work till 2000 and to [14, 15, 12, 24, 11] for a sample of papers in the last years. To mention some results explicitly, we recall that for every $r \in \mathbb{R}_{\ge 1} \cup \{\infty\}$, there is a Dedekind domain R with torsion class group such that $\rho(R) = r$ ([1]). A characterization of when the elasticity of finitely generated domains is finite is given in [23]. The elasticity of C-monoids (they include wide classes of Mori domains with nontrivial conductor) is rational or infinite ([26]).

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In [13], Chapman et al. initiated the study of the set $\{\rho(L) \mid L \in \mathcal{L}(H)\} \subset \mathbb{Q}_{\geq 1}$ of all elasticities. We say that H is *fully elastic* if for every rational number q with $1 < q < \rho(H)$ there is an $L \in \mathcal{L}(H)$ such that $\rho(L) = q$. Monoids having accepted elasticity and having a prime element are fully elastic ([9]), and it was shown only recently that all transfer Krull monoids (they include Krull domains and wide classes of non-commutative Dedekind domains) are fully elastic ([21, Theorem 3.1]). On the other hand, strongly primary monoids (including one-dimensional local Mori domains and numerical monoids) are not fully elastic ([20, Theorem 5.5]). Arithmetic congruence monoids which are not fully elastic can be found in the survey [8].

Anderson and Pruis [7] studied, for every $a \in H$, the quantities

$$\rho_*(a) = \lim_{n\to\infty} \frac{\min \mathsf{L}(a^n)}{n} \quad \text{and} \quad \rho^*(a) = \lim_{n\to\infty} \frac{\max \mathsf{L}(a^n)}{n}$$

(see also [3]) and they conjectured these invariants are rational for Krull domains and for noetherian domains. This was confirmed in [16] for Krull domains and for various classes of noetherian domains but is still open in general. In [10], Baginski et al. introduced the concept of asymptotic elasticities. For $a \in H \setminus H^\times$,

$$\overline{\rho}(a) = \lim_{n\to\infty} \rho(a^n) = \frac{\rho^*(a)}{\rho_*(a)}$$

is the *asymptotic elasticity* of a, and

$$\overline{R}(H) = \{\overline{\rho}(a) \mid a \in H \setminus H^\times\} \subset \mathbb{R}_{\geq 1} \cup \{\infty\}$$

denotes the *set of asymptotic elasticities* of H. We say that H is *asymptotic fully elastic* if for every rational number q with $\inf \overline{R}(H) < q < \rho(H)$ there is an $a \in H$ such that $\overline{\rho}(a) = q$ (note that $\sup \overline{R}(H) = \rho(H)$). Now we can formulate our main results.

Theorem 1.1. *Let H be a locally finitely generated monoid. Then H is asymptotic fully elastic and if $\inf \overline{R}(H) = 1$, then H is fully elastic.*

Every Krull monoid H is locally finitely generated with $\inf \overline{R}(H) = 1$ ([17, Proposition 2.7.8.3] and [20, Lemma 5.4]). However, there are locally finitely generated monoids H with $\inf \overline{R}(H) > 1$ that are fully elastic (Example 3.6).

Theorem 1.2. *Let H be a monoid such that H_{red} is finitely generated and let $r = \inf \overline{R}(H)$. Then*

$$\{q \in \mathbb{Q} \mid r \leq q \leq \rho(H)\} \subset \{\rho(L) \mid L \in \mathcal{L}(H)\}$$

and r is the only possible limit point of the set $\{\rho(L) \mid L \in \mathcal{L}(H)$ and $1 \leq \rho(L) < r\}$. Moreover, H is fully elastic if and only if $r = 1$.

In Section 2 we provide the required background. The proofs of Theorems 1.1 and 1.2 are given in Section 3, and then in Section 4 we apply our results to v-noetherian weakly Krull monoids.

2. Background on monoids and their arithmetic

Our notation and terminology are consistent with [17]. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $k \in \mathbb{N}$, we denote by $\mathbb{N}_{\geq k}$ the set of all integers greater than or equal to k and for $a, b \in \mathbb{Q}$, we denote by $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$ the discrete, finite interval between a and b. For a finite subset $L \subset \mathbb{N}$, we set $\rho(L) = \max L / \min L$ and $\rho(\{0\}) = 1$. For subsets $A, B \subset \mathbb{Z}$, $A + B = \{a + b \mid a \in A, b \in B\}$ denotes their sumset.

Monoids. Throughout this paper, a *monoid* means a commutative cancellative semigroup with identity element, and we use multiplicative notation. If R is a domain, then its multiplicative semigroup $R^\bullet = R \setminus \{0\}$ of nonzero elements is a monoid.

Let H be a monoid with identity element $1 = 1_H \in H$. We denote by $H^\times$ the unit group of H, by $H_{\mathrm{red}} = H/H^\times = \{aH^\times \mid a \in H\}$ the associated reduced monoid, and by $\mathsf{q}(H)$ the quotient group of H. If $H^\times = \{1\}$, we say that H is reduced. Two elements $a, b \in H$ are said to be associated if $aH^\times = bH^\times$. A submonoid $S \subset H$ is said to be

- saturated if $a, b \in S$ and $a \mid_H b$ implies that $a \mid_S b$ (equivalently, $S = \mathsf{q}(S) \cap H$).
- divisor-closed if $a \in S$, $b \in H$, and $b \mid a$ implies that $b \in S$.

If $a \in H$, then
$$[\![a]\!] = \{b \in H \mid b \mid a^n \text{ for some } n \in \mathbb{N}\} \subset H$$
is the smallest divisor-closed submonoid of H containing a. An element $u \in H$ is said to be irreducible (or an atom) if $u \notin H^\times$ and any equation of the form $u = ab$, with $a, b \in H$, implies that $a \in H^\times$ or $b \in H^\times$. Let $\mathcal{A}(H)$ denote the set of atoms. For a set $\mathcal{P}$, we denote by $\mathcal{F}(\mathcal{P})$ the *free abelian monoid* with basis $\mathcal{P}$. Then every $a \in \mathcal{F}(\mathcal{P})$ has a unique representation of the form
$$a = \prod_{p \in \mathcal{P}} p^{\mathsf{v}_p(a)} \quad \text{with} \quad \mathsf{v}_p(a) \in \mathbb{N}_0 \ \text{ and } \ \mathsf{v}_p(a) = 0 \ \text{ for almost all } \ p \in \mathcal{P},$$
and we call $|a|_{\mathcal{F}(\mathcal{P})} = |a| = \sum_{p \in \mathcal{P}} \mathsf{v}_p(a)$ the *length* of a.

The monoid H is said to be

- *atomic* if every non-unit is a finite product of atoms.
- *factorial* if it is atomic and every atom is prime.
- *finitely generated* if it has a finite generating set (equivalently, H_{red} and $H^\times$ are both finitely generated).
- *locally finitely generated* if for every $a \in H$ the monoid $[\![a]\!]_{\mathrm{red}}$ is finitely generated (equivalently, there are only finitely many atoms (up to associates) dividing some power of a).

We gather some examples of locally finitely generated monoids.

Example 2.1.

1. Clearly, every monoid H having (up to associates) only finitely many atoms that are not prime is locally finitely generated. Atomic domains, with almost all atoms being prime, are called

generalized Cohen-Kaplansky domains and they were introduced in [2]. The monoid of integral invertible ideals of a domain is finitely generated if and only if the domain is a Cohen-Kaplansky domain ([5, Theorem 4.3]).

2. Saturated submonoids (whence, in particular, divisor-closed submonoids) and coproducts of locally finitely generated monoids are locally finitely generated ([17, Proposition 2.7.8]). Thus, Krull monoids are locally finitely generated.

3. Let R be a factorial domain. Then the ring of integer-valued polynomials is (in general) not a Krull domain. But for every $f \in \mathrm{Int}(R)$, the submonoid $[\![f]\!] \subset \mathrm{Int}(R)$ is Krull ([25, Theorem 5.2]) whence $\mathrm{Int}(R)$ is locally finitely generated.

4. Let R be an order in a Dedekind domain, say $R \subset \overline{R}$, where $\overline{R}$ is the integral closure of R and $\overline{R}$ is a Dedekind domain. If the class group $\mathcal{C}(\overline{R})$ and the residue class ring $\overline{R}/(R\colon\overline{R})$ is finite, and R has finite elasticity, then R is locally finitely generated (see [17, Corollary 3.7.2] and [17, Theorem 3.7.1] for a more general result in the setting of weakly Krull domains).

Arithmetic of monoids. If $a \in H \setminus H^\times$ and $a = u_1 \cdot \ldots \cdot u_k$, where $k \in \mathbb{N}$ and $u_1, \ldots, u_k \in \mathcal{A}(H)$, then k is a factorization length of a, and

$$\mathsf{L}(a) = \{k \mid k \text{ is a factorization length of } a\} \subset \mathbb{N}$$

denotes the *set of lengths* of a. It is convenient to set $\mathsf{L}(a) = \{0\}$ for all $a \in H^\times$. The family

$$\mathcal{L}(H) = \{\mathsf{L}(a) \mid a \in H\}$$

is called the *system of sets of lengths* of H. The monoid H is said to be

- *half-factorial* if it is atomic and $|L| = 1$ for every $L \in \mathcal{L}(H)$.
- *a* BF*-monoid* if it is atomic and L is finite for every $L \in \mathcal{L}(H)$.

Clearly, every factorial monoid is half-factorial (but not conversely) and every v-noetherian monoid is a BF-monoid ([17, Theorem 2.2.9]). Let H be a BF-monoid. The *elasticity* $\rho(a)$ of an element $a \in H$ is defined as the elasticity of its set of lengths whence

$$\rho(a) = \rho(\mathsf{L}(a)) = \frac{\max \mathsf{L}(a)}{\min \mathsf{L}(a)}$$

and the supremum

$$\rho(H) = \sup\{\rho(L) \mid L \in \mathcal{L}(H)\} \in \mathbb{R}_{\ge 1} \cup \{\infty\}$$

denotes the *elasticity* of H. The monoid H

- has *accepted elasticity* if there is some $L \in \mathcal{L}(H)$ such that $\rho(L) = \rho(H)$.
- is *fully elastic* if for every $q \in \mathbb{Q}$ with $1 < q < \rho(H)$ there is some $L \in \mathcal{L}(H)$ such that $\rho(L) = q$.

The *asymptotic elasticity* $\overline{\rho}(a)$ of an element $a \in H \setminus H^\times$ is defined as

$$\overline{\rho}(a) = \lim_{n \to \infty} \rho(a^n) \qquad \text{(note that the limit exists by [17, Theorem 3.8.1])}\,,$$

and

$$\overline{R}(H) = \{\overline{\rho}(a) \mid a \in H \setminus H^\times\} \subset \mathbb{R}_{\ge 1} \cup \{\infty\}$$

denotes the *set of asymptotic elasticities.* We conclude this section with a technical lemma.

Lemma 2.2. *Let $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \mathbb{N}$ be positive integers. If there exist $x_1, \ldots, x_n \in \mathbb{N}_0$ and $t \ge 2$ such that $a_1x_1 + \ldots + a_nx_n = ta_1 \cdot \ldots \cdot a_n$, then there exist $x_i' \in [0, x_i]$, for all $i \in [1, n]$, such that*

$$a_1x_1' + \ldots + a_nx_n' = a_1 \cdot \ldots \cdot a_n\,.$$

In particular, there exist $x_i^{(j)} \in [0, x_i]$, for all $i \in [1, n]$ and $j \in [1, t]$, such that $\sum_{j \in [1,t]} x_i^{(j)} = x_i$ for every $i \in [1, n]$ and $\sum_{i \in [1,n]} a_i x_i^{(j)} = a_1 \cdot \ldots \cdot a_n$ for every $j \in [1, t]$.

Proof. The assertion is obvious for $n = 1$. If $n = 2$, then $a_1x_1 \ge a_1a_2$ or $a_2x_2 \ge a_1a_2$ whence the assertion follows immediately. Suppose that $n \ge 3$ and distinguish two cases.

Case 1. $\min\{a_1, \ldots, a_n\} \ge 2$.

After renumbering if necessary we assume that $a_1x_1 \ge \frac{ta_1 \cdot \ldots \cdot a_n}{n}$. Then

$$x_1 \ge \frac{2a_2 \cdot \ldots \cdot a_n}{n} \ge \frac{2^{n-1}\min\{a_2, \ldots, a_n\}}{n} \ge \min\{a_2, \ldots, a_n\}\,. \tag{2.1}$$

For each $i \in [2, n]$, we set $x_i = y_ia_1 + r_i$ with $r_i \in [0, a_1 - 1]$. Then

$$a_2r_2 + \ldots + a_nr_n \le (a_1 - 1)(a_2 + \ldots + a_n) \le a_1 \cdot \ldots \cdot a_n\,.$$

Therefore $a_1(x_1 + y_2a_2 + \ldots + y_na_n) \ge ta_1 \cdot \ldots \cdot a_n - a_1 \cdot \ldots \cdot a_n \ge a_1 \cdot \ldots \cdot a_n$ which implies that

$$x_1 + y_2a_2 + \ldots + y_na_n \ge a_2 \cdot \ldots \cdot a_n\,.$$

If $y_2a_2 + \ldots + y_na_n \le a_2 \cdot \ldots \cdot a_n$, then there exists $x_1' \in [0, x_1]$ such that $x_1' + y_2a_2 + \ldots + y_na_n = a_2 \cdot \ldots \cdot a_n$ and hence $a_1x_1' + a_2y_2a_1 + \ldots + a_ny_na_1 = a_1 \cdot \ldots \cdot a_n$. If $y_2a_2 + \ldots + y_na_n > a_2 \cdot \ldots \cdot a_n$, then we can choose $y_i' \in [0, y_i]$ for each $i \in [2, n]$ such that

$$0 \le a_2 \ldots a_n - (y_2'a_2 + \ldots + y_n'a_n) \le \min\{a_2, \ldots, a_n\} \le x_1\,.$$

Thus there exists $x_1' \in [0, x_1]$ such that $x_1' + y_2'a_2 + \ldots y_n'a_n = a_2 \cdot \ldots \cdot a_n$ and hence $a_1x_1' + a_2y_2'a_1 + \ldots + a_ny_n'a_1 = a_1 \cdot \ldots \cdot a_n$.

Case 2. $\min\{a_1, \ldots, a_n\} = 1$.

After renumbering if necessary we assume that there exists $\tau \in [1, n]$ such that $a_\tau = a_{\tau+1} = \ldots = a_n = 1$ and $a_i \ge 2$ for all $i \in [1, \tau - 1]$. If $x_\tau + \ldots + x_n \ge a_1 \cdot \ldots \cdot a_n$ or $\tau \le 2$, then the assertion follows immediately. Suppose $\tau = 3$. Then $a_1x_1 + a_2x_2 + x_3 + \ldots + x_n = ta_1a_2 \ge 2a_1a_2$. If $a_1x_1 \ge a_1a_2$ or $a_2a_3 \ge a_2x_3$, then we are done. Otherwise $a_1x_1 < a_1a_2$ and

$a_1x_1+x_3+\ldots+x_n \ge a_1a_2$. We choose $x_i' \in [0, x_i]$ for all $i \in [3, n]$ such that $\sum_{i=3}^n x_i' = a_1a_2 - a_1x_1$. Then $a_1x_2 + \sum_{i=3}^n x_i' = a_1a_2 = a_1 \cdot \ldots \cdot a_n$.

Now we assume that $\tau \ge 3$ and $x_\tau + \ldots + x_n < a_1 \cdot \ldots \cdot a_n$

Case 2.1. $\frac{ta_1\cdot\ldots\cdot a_n}{\tau} \le x_\tau + \ldots + x_n < a_1 \cdot \ldots \cdot a_n$.

Thus

$$x_\tau + \ldots + x_n \ge \frac{2a_1 \ldots a_\tau}{\tau} \ge \frac{2^{\tau-1}\min\{a_1, \ldots, a_{\tau-1}\}}{\tau} \ge \min\{a_1, \ldots, a_{\tau-1}\}\,.$$

$$\text{and} \quad a_1x_1 + \ldots + a_{\tau-1}x_{\tau-1} > a_1 \cdot \ldots \cdot a_n\,.$$

We choose $x_i' \in [0, x_i]$ for each $i \in [1, \tau-1]$ such that

$$0 \le a_1 \cdot \ldots \cdot a_n - (a_1x_1' + \ldots + a_{\tau-1}x_{\tau-1}') \le \min\{a_1, \ldots, a_{\tau-1}\} \le x_\tau + \ldots + x_n\,.$$

Therefore we choose $x_i' \in [0, x_i]$ for each $i \in [\tau+1, n]$ such that $x_\tau' + \ldots + x_n' = a_1 \cdot \ldots \cdot a_n - (a_1x_1' + \ldots + a_{\tau-1}x_{\tau-1}')$. It follows that $a_1x_1' + \ldots + a_nx_n' = a_1 \cdot \ldots \cdot a_n$.

Case 2.2. $\frac{ta_1\cdot\ldots\cdot a_n}{\tau} > x_\tau + \ldots + x_n$.

Then there must exist $j \in [1, \tau-1]$ such that $a_jx_j \ge \frac{ta_1\cdot\ldots\cdot a_n}{\tau}$, say $a_1x_1 \ge \frac{ta_1\cdot\ldots\cdot a_n}{\tau}$, which implies that

$$x_1 \ge \frac{ta_2 \ldots a_{\tau-1}}{\tau} \ge \frac{2^{\tau-2}\min\{a_1, \ldots, a_{\tau-1}\}}{\tau} \ge \min\{a_1, \ldots, a_{\tau-1}\}\,.$$

For each $i \in [2, \tau-1]$, we set $x_i = y_ia_1 + r_i$ with $r_i \in [0, a_1-1]$. Then

$$a_2r_2 + \ldots + a_{\tau-1}r_{\tau-1} \le (a_1-1)(a_2 + \ldots + a_{\tau-1}) \le a_1 \cdot \ldots \cdot a_{\tau-1} = a_1 \cdot \ldots \cdot a_n\,.$$

Therefore

$$a_1(x_1 + y_2a_2 + \ldots + y_{\tau-1}a_{\tau-1}) + x_\tau + \ldots + x_n \ge ta_1 \cdot \ldots \cdot a_n - a_1 \cdot \ldots \cdot a_n \ge a_1 \cdot \ldots \cdot a_n\,.$$

Suppose that $a_1(x_1 + y_2a_2 + \ldots + y_{\tau-1}a_{\tau-1}) < a_1 \cdot \ldots \cdot a_n$. Then we can choose $x_1' = x_1$, $x_i' = a_1y_i$ for all $i \in [2, \tau-1]$, and $x_i' \in [0, x_i]$ for all $i \in [\tau, n]$ with

$$\sum_{i=\tau}^n x_i' = a_1 \cdot \ldots \cdot a_n - a_1(x_1 + y_2a_2 + \ldots + y_{\tau-1}a_{\tau-1}) \in [1, x_\tau + \ldots + x_n]\,.$$

It follows that $a_1x_1' + \ldots a_nx_n' = a_1 \cdot \ldots \cdot a_n$.

Suppose that $a_1(x_1+y_2a_2+\ldots+y_{\tau-1}a_{\tau-1}) \ge a_1\cdot\ldots\cdot a_n$. If $y_2a_2+\ldots+y_{\tau-1}a_{\tau-1} \le a_2\cdot\ldots\cdot a_n$, then there exists $x_1' \in [0, x_1]$ such that $x_1' + y_2a_2 + \ldots + y_{\tau-1}a_{\tau-1} = a_2 \cdot \ldots \cdot a_n$ and hence $a_1x_1' + a_2y_2a_1 + \ldots + a_{\tau-1}y_{\tau-1}a_1 = a_1 \cdot \ldots \cdot a_n$. If $y_2a_2 + \ldots + y_{\tau-1}a_{\tau-1} > a_2 \cdot \ldots \cdot a_n$, then we can choose $y_i' \in [0, y_i]$ for each $i \in [2, \tau-1]$ such that

$$0 \le a_2 \cdot \ldots \cdot a_n - (y_2'a_2 + \ldots + y_{\tau-1}'a_{\tau-1}) \le \min\{a_2, \ldots, a_{\tau-1}\} \le x_1\,.$$

Thus there exists $x_1' \in [0, x_1]$ such that $x_1' + y_2'a_2 + \ldots + y_{\tau-1}'a_{\tau-1} = a_2 \cdot \ldots \cdot a_n$ and hence $a_1x_1' + a_2y_2'a_1 + \ldots + a_{\tau-1}y_{\tau-1}'a_1 = a_1 \cdot \ldots \cdot a_n$.

The in particular statement follows by induction on t. $\square$

3. Locally finitely generated monoids

In this section we prove Theorems 1.1 and 1.2 formulated in the Introduction. We start with a series of lemmas.

Lemma 3.1. *Let H be a monoid such that H_{red} is finitely generated.*

1. *For every $a \in H \setminus H^\times$, there exists $N = N(a) \in \mathbb{N}$ such that $\rho(a^{tN}) = \rho(a^N)$ for every $t \in \mathbb{N}$ and $\overline{\rho}(a) = \rho(a^N)$.*
2. *$\overline{R}(H) = \{\rho(a) \mid a \in H \setminus H^\times$ with the property that $\rho(a^t) = \rho(a)$ for all $t \in \mathbb{N}\}$.*
3. *There exists $a \in H$ such that $\rho(a) = \rho(H) = \sup \overline{R}(H)$.*
4. *There exists an atom $b \in \mathcal{A}(H)$ such that $\lim_{n\to\infty} \rho(b^n) = \inf \overline{R}(H)$.*

Proof. 1. This follows immediately by [17, Theorem 3.8.1].

2. This follows from the definition and from 1.

3. It follows from [17, Theorem 3.1.4] that H has accepted elasticity. Let $a \in H \setminus H^\times$ with $\rho(a) = \rho(H)$. Then $\lim_{n\to\infty} \rho(a^n) = \rho(H) = \sup \overline{R}(H)$.

4. If $a \in H \setminus H^\times$ is not an atom, then $a = u_1 \cdot \ldots \cdot u_\ell$ where $\ell \ge 2$ and $u_i \in \mathcal{A}(H)$. Let $N \in \mathbb{N}$ such that $\lim_{n\to\infty} \rho(a^n) = \rho(a^N)$ and $\lim_{n\to\infty} \rho(u_i^n) = \rho(u_i^N)$ for every $i \in [1,\ell]$. Then $\rho(a^N) \ge \min\{\rho(u_i^N)] \mid i \in [1,\ell]\}$. It follows by $\mathcal{A}(H)$ is finite that

$$\inf \overline{R}(H) = \inf \left\{ \lim_{n\to\infty} \rho(u^n) \mid u \in \mathcal{A}(H) \right\} = \min \left\{ \lim_{n\to\infty} \rho(u^n) \mid u \in \mathcal{A}(H) \right\}. \quad \square$$

Definition 3.2. Let H be an atomic monoid and let $a, b \in H \setminus H^\times$. We say the pair $(k,\ell) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$ is nice with respect to (a,b) if for every $t \in \mathbb{N}$

$$\max \mathsf{L}((a^k b^\ell)^t) = t \max \mathsf{L}(a^k b^\ell) \quad \text{and} \quad \min \mathsf{L}((a^k b^\ell)^t) = t \min \mathsf{L}(a^k b^\ell).$$

It is easy to check that (k,ℓ) is nice if and only if $\rho((a^k b^\ell)^t) = \rho(a^k b^\ell)$ for all $t \in \mathbb{N}$.

Lemma 3.3. *Let H be an atomic monoid, $a, b \in H \setminus H^\times$, and let $(k,\ell) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$ be a nice pair with respect to (a,b).*

1. *$(tk, t\ell)$ is a nice pair with respect to (a,b) for each $t \in \mathbb{N}$.*
2. *$\rho(a^k b^\ell) \in \overline{R}(H)$.*
3. *For every $x \in \mathbb{Q}$ with $x \ge 0$, there exists a nice pair (k',ℓ') with respect to (a,b) such that $\ell'/k' = x$, where $k' \in \mathbb{N}$ and $\ell' \in \mathbb{N}_0$.*

Proof. 1. It is obvious by definition.

2. It follows immediately by Lemma 3.1.2.

3. Let $x \in \mathbb{Q}$ with $x \ge 0$ and let $m, n \in \mathbb{N}_0$ such that $x = m/n$. Then $a^m b^n \in H \setminus H^\times$. It follows by Lemma 3.1.1 that there exists $N \in \mathbb{N}$ such that $\rho((a^{Nm} b^{Nn})^t) = \rho(a^{Nm} b^{Nn})$ for all $t \in \mathbb{N}$. Therefore (Nm, Nn) is a nice pair with respect to (a,b) such that $Nm/Nn = x$. $\square$

Proposition 3.4. *Let H be a monoid such that H_{red} is finitely generated and $\min \overline{R}(H) < \max \overline{R}(H)$. Let $c \in H$ with $\lim_{n\to\infty} \rho(c^n) = \max \overline{R}(H)$ and let $b \in \mathcal{A}(H)$ with $\lim_{n\to\infty} \rho(b^n) = \min \overline{R}(H)$.*

1. *There exists $M \in \mathbb{N}$ satisfying that for every $k \in \mathbb{N}$ and every $\ell \in \mathbb{N}_0$, there exist $\ell_1, \dots, \ell_k \in \mathbb{Z}$ with $\ell_1 + \dots + \ell_k = \ell$ and all $c^M b^{\ell_i} \in H$ such that*
$$\max \mathsf{L}(c^{kM} b^{\ell}) = \sum_{i=1}^{k} \max \mathsf{L}(c^M b^{\ell_i}) \quad \text{and} \quad \min \mathsf{L}(c^{kM} b^{\ell}) = \sum_{i=1}^{k} \min \mathsf{L}(c^M b^{\ell_i})\,.$$
2. *Set $a = c^M$. Let $(k, \ell) \in \mathbb{N}_0^2 \setminus \{(0,0)\}$ be a nice pair with respect to (a, b) and let $\ell_1, \dots, \ell_k \in \mathbb{Z}$ with $\ell_1 + \dots + \ell_k = \ell$ and all $ab^{\ell_i} \in H$ such that*
$$\max \mathsf{L}(a^k b^{\ell}) = \sum_{i=1}^{k} \max \mathsf{L}(ab^{\ell_i}) \text{ and } \min \mathsf{L}(a^k b^{\ell}) = \sum_{i=1}^{k} \min \mathsf{L}(ab^{\ell_i})\,.$$
   (a) *For distinct $i_1, i_2 \in [1, k]$ and any $t_1, t_2 \in \mathbb{N}_0$, we have*
$$\begin{aligned} & \max \mathsf{L}((ab^{\ell_{i_1}})^{t_1}(ab^{\ell_{i_2}})^{t_2}) = t_1 \max \mathsf{L}(ab^{\ell_{i_1}}) + t_2 \max \mathsf{L}(ab^{\ell_{i_2}})\,, \\ \text{and} \quad & \min \mathsf{L}((ab^{\ell_{i_1}})^{t_1}(ab^{\ell_{i_2}})^{t_2}) = t_1 \min \mathsf{L}(ab^{\ell_{i_1}}) + t_2 \min \mathsf{L}(ab^{\ell_{i_2}})\,. \end{aligned}$$
   In particular, $(1, \ell_i)$ is a nice pair with respect to (a, b) for each $i \in [1, k]$.

   (b) *Let τ be the maximal non-negative integer ℓ such that $\rho(\mathsf{L}(ab^{\ell})) = \rho(\mathsf{L}(a))$. If $\ell/k > \tau$, then $\ell_j \ge \tau$ for every $j \in [1, k]$.*
3. *Let I be the set of all $t \in \mathbb{N}_{\ge \tau}$ such that $(1, t)$ is a nice pair respect to (a, b).*

   (a) *I is infinite.*

   (b) *Suppose $I = \{t_1, t_2, \dots\} \subset \mathbb{N}_{\ge \tau}$ with $\tau = t_1 < t_2 < \dots$. Then $\rho(ab^{t_j}) \ge \rho(ab^{t_{j+1}})$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} \rho(ab^{t_j}) = \inf \overline{R}(H)$.*
4. *Let $i \in \mathbb{N}$, let $x \in \mathbb{Q}$ with $t_i < x < t_{i+1}$, and let $k, \ell \in \mathbb{N}$ with $\ell/k = x$ be such that (k, ℓ) is a nice pair with respect to (a, b). Then*
$$\rho(a^k b^{\ell}) = \frac{(t_{i+1} - x) \max \mathsf{L}(ab^{t_{i+1}}) + (x - t_i) \max \mathsf{L}(ab^{t_i})}{(t_{i+1} - x) \min \mathsf{L}(ab^{t_{i+1}}) + (x - t_i) \min \mathsf{L}(ab^{t_i})}\,.$$
5. *$\overline{R}(H) = \{q \in \mathbb{Q} \mid \inf \overline{R}(H) \le q \le \sup \overline{R}(H)\}$.*

Proof. We may assume that H is reduced. We denote by $\mathsf{Z}(H) := \mathcal{F}(\mathcal{A}(H))$ the factorization monoid of H and by $\pi : \mathsf{Z}(H) \to H$ the factorization homomorphism. If $z \in \mathsf{Z}(H)$, then $|z| = |z|_{\mathcal{F}(\mathcal{A}(H))}$ denotes the length of z.

1. Assume to the contrary that there exists $n \in \mathbb{N}$ such that $c \,|\, b^n$. Then $b^n = cd$ for some $d \in H \setminus H^{\times}$. Let $N \in \mathbb{N}$ such that $\lim_{m\to\infty} \rho(c^m) = \rho(c^N)$, $\lim_{m\to\infty} \rho(b^m) = \rho(b^{nN})$, and

$\lim_{m\to\infty}\rho(d^m)=\rho(d^N)$. Thus $\rho(b^{nN})\ge\min\{\rho(d^N),\rho(c^N)\}\ge\rho(d^N)\ge\rho(b^{nN})$ which implies that $\rho(b^{nN})=\rho(d^N)=\rho(c^N)$, a contradiction. Thus $c\nmid b^n$ for every $n\in\mathbb{N}$.

Let

$$H_0=\{(x,y)\in\mathsf{Z}(H)\times\mathsf{Z}(H)\mid \pi(x)=\pi(y)=c^kb^\ell\in H \text{ for some } k\in\mathbb{N}_0 \text{ and some } \ell\in\mathbb{Z}\}\,.$$

Then H_0 is a saturated submonoid of $\mathsf{Z}(H)\times\mathsf{Z}(H)$. Since H is finitely generated, it follows by [17, Proposition 2.7.5] that H_0 is finitely generated.

Suppose $\mathcal{A}(H_0)=\{(x_1,y_1),\ldots,(x_t,y_t)\}$ with $\pi(x_i)=\pi(y_i)=c^{k_i}b^{\ell_i}\in H$ for each $i\in[1,t]$, where $t\in\mathbb{N},k_1,\ldots,k_t\in\mathbb{N}_0$ and $\ell_1,\ldots,\ell_t\in\mathbb{Z}$. Note that $(b,b)\in H_0$ is an atom of H_0. We obtain that $\min\{k_1,\ldots,k_t\}=0$. After renumbering if necessary, we assume that there exists $t_0\in[1,t]$ such that $k_i\ge1$ for all $i\in[1,t_0]$ and $k_i=0$ for all $i\in[t_0,t]$.

Let $M=\prod_{i=1}^{t_0}k_i$ and let $k\in\mathbb{N}$, $\ell\in\mathbb{N}_0$. We choose $(x,y)\in\mathsf{Z}(H)\times\mathsf{Z}(H)$ with $\pi(x)=\pi(y)=c^{kM}b^\ell$ such that $|x|=\min\mathsf{L}(c^{kM}b^\ell)$ and $|y|=\max\mathsf{L}(c^{kM}b^\ell)$.

Suppose $(x,y)=\prod_{i=1}^t(x_i,y_i)^{v_i}$, where $v_i\in\mathbb{N}_0$. If $v_i\ne0$, then $|x_i|=\min\mathsf{L}(c^{k_i}b^{\ell_i})$ and $|y_i|=\max\mathsf{L}(c^{k_i}b^{\ell_i})$. Therefore

$$\begin{aligned}\max\mathsf{L}(c^{kM}b^\ell)&=\sum_{i=1}^t v_i|y_i|=\sum_{i=1}^t v_i\max\mathsf{L}(c^{k_i}b^{\ell_i})\,,\\ \min\mathsf{L}(c^{kM}b^\ell)&=\sum_{i=1}^t v_i|x_i|=\sum_{i=1}^t v_i\min\mathsf{L}(c^{k_i}b^{\ell_i})\,,\\ \sum_{i=1}^t k_iv_i&=kM \quad\text{and}\quad \sum_{i=1}^t \ell_iv_i=\ell\,.\end{aligned}$$

Since

$$kM=k\prod_{i=1}^{t_0}k_i=\sum_{i=1}^t v_ik_i=\sum_{i=1}^{t_0}v_ik_i\,,$$

it follows by Lemma 2.2 that there exist $x_i^{(j)}\in[0,v_i],i\in[1,t_0],j\in[1,k]$ such that $\sum_{j=1}^k x_i^{(j)}=v_i$ for every $i\in[1,t_0]$ and $\sum_{i=1}^{t_0}x_i^{(j)}k_i=M$ for every $j\in[1,k]$. Let

$$\begin{aligned}\ell_1'&=\sum_{i=1}^{t_0}x_i^{(1)}\ell_i+\sum_{i=t_0+1}^t v_i\ell_i\\ \text{and}\quad \ell_j'&=\sum_{i=1}^{t_0}x_i^{(j)}\ell_i\quad\text{for every } j\in[2,k]\,.\end{aligned}$$

Then $\ell_1'+\ldots+\ell_k'=\sum_{i=1}^t v_i\ell_i=\ell$ and

$$\begin{aligned}\max\mathsf{L}(c^{kM}b^\ell)&\ge\sum_{j=1}^k\max\mathsf{L}(c^Mb^{\ell_j'})\ge\sum_{i=1}^t v_i\max\mathsf{L}(c^{k_i}b^{\ell_i})=\max\mathsf{L}(c^{kM}b^\ell)\,,\\ \min\mathsf{L}(c^{kM}b^\ell)&\le\sum_{j=1}^k\min\mathsf{L}(c^Mb^{\ell_j'})\le\sum_{i=1}^t v_i\max\mathsf{L}(c^{k_i}b^{\ell_i})=\min\mathsf{L}(c^{kM}b^\ell)\,.\end{aligned}$$

2. (a) Without loss of generality, we assume that $i_1 = 1$ and $i_2 = 2$. Let $t = \max\{t_1, t_2\}$. Then

$$\begin{aligned}
&t\sum_{j=1}^{k} \max \mathsf{L}(ab^{\ell_i}) = \max \mathsf{L}((a^k b^\ell)^t) \\
&\ge \max \mathsf{L}\big((ab^{\ell_1})^{t_1}(ab^{\ell_2})^{t_2}\big) + (t-t_1)\max \mathsf{L}(ab^{\ell_1}) + (t-t_2)\max \mathsf{L}(ab^{\ell_2}) + t\sum_{j=3}^{k}\max \mathsf{L}(ab^{\ell_i}) \\
&\ge t\sum_{j=1}^{k}\max \mathsf{L}(ab^{\ell_i})\,.
\end{aligned}$$

It follows that

$$\max \mathsf{L}\big((ab^{\ell_1})^{t_1}(ab^{\ell_2})^{t_2}\big) = t_1 \max \mathsf{L}(ab^{\ell_1}) + t_2 \max \mathsf{L}(ab^{\ell_2})\,.$$

By the similar argument, we can obtain

$$\min \mathsf{L}\big((ab^{\ell_1})^{t_1}(ab^{\ell_2})^{t_2}\big) = t_1 \min \mathsf{L}(ab^{\ell_1}) + t_2 \min \mathsf{L}(ab^{\ell_2})\,.$$

(b) Suppose $\ell/k > \tau$. Since $\sum_{i=1}^{k} \ell_i = \ell > \tau$, there exists some $i \in [1,k]$, say $i = 1$, such that $\ell_1 > \tau$. Thus $\rho(\mathsf{L}(ab^{\ell_1})) < \rho(\mathsf{L}(a))$ by the definition of τ.

Assume to the contrary that there exists some $i \in [1,k]$, say $i = 2$, such that $\ell_2 < \tau$. Then

$$\begin{aligned}
\rho(\mathsf{L}(a)) &= \rho(\mathsf{L}(ab^\tau)) = \rho(\mathsf{L}((ab^\tau)^{\ell_1-\ell_2})) \\
&= \frac{\max \mathsf{L}(a^{(\ell_1-\ell_2)}b^{\tau(\ell_1-\ell_2)})}{\min \mathsf{L}(a^{(\ell_1-\ell_2)}b^{\tau(\ell_1-\ell_2)})} \\
&= \frac{\max \mathsf{L}\big((ab^{\ell_1})^{\tau-\ell_2}(ab^{\ell_2})^{\ell_1-\tau}\big)}{\min \mathsf{L}\big((ab^{\ell_1})^{\tau-\ell_2}(ab^{\ell_2})^{\ell_1-\tau}\big)} \\
&= \frac{(\ell_1-\tau)\max \mathsf{L}(ab^{\ell_2}) + (\tau-\ell_2)\max \mathsf{L}(ab^{\ell_1})}{(\ell_1-\tau)\min \mathsf{L}(ab^{\ell_2}) + (\tau-\ell_2)\min \mathsf{L}(ab^{\ell_1})} \\
&< \rho(\mathsf{L}(a))\,,
\end{aligned}$$

a contradiction.

3. (a) Assume to the contrary that I is finite. Let $r = \max I$ and let $k, \ell \in \mathbb{N}$ with $\ell/k > r \ge \tau$ such that (k,ℓ) is a nice pair with respect to (a,b). Then there exist $\ell_1, \ell_2, \ldots, \ell_k \in \mathbb{Z}$ with $\ell_1 + \ldots + \ell_k = \ell$ and

$$\max \mathsf{L}(a^k b^\ell) = \sum_{i=1}^{k} \max \mathsf{L}(ab^{\ell_i}) \quad \text{and} \quad \min \mathsf{L}(a^k b^\ell) = \sum_{i=1}^{k} \min \mathsf{L}(ab^{\ell_i})\,.$$

It follows by Lemma 3.4.2 that $\ell_i \ge \tau$ and $\ell_i \in I$ for all $i \in [1,k]$. Since $\sum_{i=1}^{k} \ell_i = \ell > kr$, there exists some $i \in [1,k]$, say $i = 1$, such that $\ell_1 > r$. We obtain a contradiction to $\ell_1 \in I$ and $\max I = r$.

(b) Let $I=\{t_1,t_2,\ldots\}$ with $\tau=t_0<t_1<\ldots$. For every $i\in\mathbb{N}$, we have that

$$\begin{aligned}\rho(\mathsf{L}(ab^{t_i}))&=\rho(\mathsf{L}((ab^{t_i})^{t_{i+1}-t_1}))=\rho(\mathsf{L}((ab^{t_1})^{t_{i+1}-t_i}(ab^{t_{i+1}})^{t_i-t_1}))\\&\ge\frac{(t_{i+1}-t_i)\max\mathsf{L}(ab^{t_1})+(t_i-t_1)\max\mathsf{L}(ab^{t_{i+1}})}{(t_{i+1}-t_i)\min\mathsf{L}(ab^{t_1})+(t_i-t_1)\min\mathsf{L}(ab^{t_{i+1}})}\\&\ge\min\{\rho(\mathsf{L}(ab^{t_1})),\rho(\mathsf{L}(ab^{t_{i+1}}))\}\\&=\rho(\mathsf{L}(ab^{t_{i+1}}))\,.\end{aligned}$$

In order to show $\lim_{j\to\infty}\rho(ab^{t_j})=\inf\overline{R}(H)$, we prove that $\lim_{\ell\to\infty}\rho(ab^\ell)=\inf\overline{R}(H)$.

Note that $a\nmid b^n$ for any $n\in\mathbb{N}$. Let

$$H_1=\{(x,y)\in\mathsf{Z}(H)\times\mathsf{Z}(H)\mid\pi(x)=\pi(y)=a^kb^\ell\in H\text{ for some }k\in\mathbb{N}_0\text{ and some }\ell\in\mathbb{Z}\}\,.$$

Then H_1 is a saturated submonoid of $\mathsf{Z}(H)\times\mathsf{Z}(H)$. Since H is finitely generated, we obtain H_1 is finitely generated. Suppose $\mathcal{A}(H_1)=\{(x_1,y_1),\ldots,(x_t,y_t)\}$ with $\pi(x_i)=\pi(y_i)=a^{k_i}b^{\ell_i}\in H$ for each $i\in[1,t]$, where $t\in\mathbb{N},k_1,\ldots,k_t\in\mathbb{N}_0$ and $\ell_1,\ldots,\ell_t\in\mathbb{Z}$. Since $(z_1,z_2)\in\mathcal{A}(H_1)$ for any $z_1,z_2\in\mathsf{z}(a)$, we have that there exists $i\in[1,t]$ such that $k_i=1$. Let $\ell_{\max}=\max\{\ell_i\mid k_i=1\}$ and $\ell_{\min}=\min\{\ell_i\mid k_i=1\}$.

For every $\ell\ge\ell_{\max}$, we let $(x,y)\in H_1$ with $\pi(x)=ab^\ell$ such that $|x|=\min\mathsf{L}(ab^\ell)$ and $|y|=\max\mathsf{L}(ab^\ell)$. Suppose $(x,y)=\prod_{i=1}^t(x_i,y_i)^{v_i}$, where all $v_i\in\mathbb{N}_0$. Therefore $1=\sum_{i=1}^t k_iv_i$ and $\ell=\ell_iv_i$. It follows that there exists $\ell'\in[\ell_{\min},\ell_{\max}]$ such that

$$\begin{aligned}\max\mathsf{L}(ab^\ell)&=\max\mathsf{L}(ab^{\ell'})+\max\mathsf{L}(b^{\ell-\ell'})\le\max\mathsf{L}(ab^{\ell_{\max}})+\max\mathsf{L}(b^{\ell-\ell_{\min}})\\\min\mathsf{L}(ab^\ell)&=\min\mathsf{L}(ab^{\ell'})+\min\mathsf{L}(b^{\ell-\ell'})\ge\min\mathsf{L}(ab^{\ell_{\min}})+\min\mathsf{L}(b^{\ell-\ell_{\max}})\\&\ge\min\mathsf{L}(ab^{\ell_{\min}})+\min\mathsf{L}(b^{\ell-\ell_{\min}})-\min\mathsf{L}(b^{\ell_{\max}-\ell_{\min}})\end{aligned}$$

If follows that

$$\rho(b^\ell)\le\rho(ab^\ell)\le\frac{\max\mathsf{L}(ab^{\ell_{max}})+\max\mathsf{L}(b^{\ell-\ell_{min}})}{\min\mathsf{L}(ab^{\ell_{min}})-\min\mathsf{L}(b^{\ell_{max}-\ell_{min}})+\min\mathsf{L}(b^{\ell-\ell_{min}})}\,.$$

Since

$$\lim_{\ell\to\infty}\frac{\max\mathsf{L}(ab^{\ell_{max}})+\max\mathsf{L}(b^{\ell-\ell_{min}})}{\min\mathsf{L}(ab^{\ell_{min}})-\min\mathsf{L}(b^{\ell_{max}-\ell_{min}})+\min\mathsf{L}(b^{\ell-\ell_{min}})}=\lim_{\ell\to\infty}\frac{\max\mathsf{L}(b^{\ell-\ell_{min}})}{\min\mathsf{L}(b^{\ell-\ell_{min}})}=\inf\overline{R}(H)\,,$$

we obtain that $\lim_{\ell\to\infty}\rho(ab^\ell)=\inf\overline{R}(H)$.

4. Let $j\in\mathbb{N}$ and $x\in\mathbb{Q}$ with $t_j<x<t_{j+1}$. There exist $k,\ell\in\mathbb{N}$ with $\ell/k=x$ such that (k,ℓ) is a nice pair with respect to (a,b). We fix such a pair. Then for any $t\in\mathbb{N}$, $(tk,t\ell)$ is a nice pair with respect to (a,b). By 1. and 2.b., we choose $\ell_1,\ldots,\ell_{tk}\in\mathbb{N}_{\ge\tau}$ with $\sum_{i=1}^{tk}\ell_i=t\ell$ and

$$\max\mathsf{L}(a^{tk}b^{t\ell})=\sum_{i=1}^{tk}\max\mathsf{L}(ab^{\ell_i})\ \text{ and }\ \min\mathsf{L}(a^{tk}b^{t\ell})=\sum_{i=1}^{tk}\min\mathsf{L}(ab^{\ell_i})\,,$$

such that

$$C_t=\big|I\cap[\min\{\ell_i\mid i\in[1,tk]\},\max\{\ell_i\mid i\in[1,tk]\}]\big|$$

is minimal. Let $N \in \mathbb{N}$ such that $C_N = \min\{C_t \mid t \in \mathbb{N}\}$ and $\ell_1, \ldots, \ell_{Nk}$ are the corresponding non-negative integers. It follows by Lemma 3.4.3.a that $\ell_i \in I$ for all $i \in [1, Nk]$.

Without loss of generality, we let $\ell_1 = \max\{\ell_i \mid i \in [1, Nk]\}$ and $\ell_2 = \min\{\ell_i \mid i \in [1, Nk]\}$. Assume to the contrary that $C_N \ge 3$. Then there is a $y \in I$ such that $\ell_2 < y < \ell_1$. Let $S = \ell_1 \cdot \ldots \cdot \ell_{Nk}$ be the sequence over $\mathbb{N}_{\ge \tau_M}$. If $\mathsf{v}_{\ell_1}(S)(\ell_1 - y) \le \mathsf{v}_{\ell_2}(S)(y - \ell_2)$, then we choose $\ell_1' \cdot \ldots \cdot \ell_{Nk(y-\ell_2)}' = \ell_2^{(y-\ell_2)\mathsf{v}_{\ell_2}(S) - \mathsf{v}_{\ell_1}(S)(\ell_1 - y)} y^{(\ell_1-\ell_2)\mathsf{v}_{\ell_1}(S)} \prod_{i=3}^{Nk} \ell_i^{y-\ell_2}$ and hence $\sum_{i=1}^{Nk(y-\ell_2)} \ell_i' = N\ell(y-\ell_2)$. Since

$$\begin{aligned}
&(\ell_1 - \ell_2) \max \mathsf{L}(ab^y) = \max \mathsf{L}((ab^y)^{\ell_1 - \ell_2}) = \\
&\qquad \max \mathsf{L}((ab^{\ell_1})^{y-\ell_2}(ab^{\ell_2})^{\ell_1 - y}) = (y - \ell_2) \max \mathsf{L}(ab^{\ell_1}) + (\ell_1 - y) \max \mathsf{L}(ab^{\ell_2})
\end{aligned}$$

and

$$\begin{aligned}
&(\ell_1 - \ell_2) \min \mathsf{L}(ab^y) = \min \mathsf{L}((ab^y)^{\ell_1 - \ell_2}) = \\
&\qquad \min \mathsf{L}((ab^{\ell_1})^{y-\ell_2}(ab^{\ell_2})^{\ell_1 - y}) = (y - \ell_2) \min \mathsf{L}(ab^{\ell_1}) + (\ell_1 - y) \min \mathsf{L}(ab^{\ell_2})\,,
\end{aligned}$$

we have

$$\begin{aligned}
\max \mathsf{L}(a^{Nk(y-\ell_2)} b^{N\ell(y-\ell_2)}) &= (y-\ell_2) \sum_{i=1}^{Nk} \max \mathsf{L}(ab^{\ell_i}) = \sum_{i=1}^{Nk(y-\ell_2)} \max \mathsf{L}(ab^{\ell_i'}) \\
\min \mathsf{L}(a^{Nk(y-\ell_2)} b^{N\ell(y-\ell_2)}) &= (y-\ell_2) \sum_{i=1}^{Nk} \min \mathsf{L}(ab^{\ell_i}) = \sum_{i=1}^{Nk(y-\ell_2)} \min \mathsf{L}(ab^{\ell_i'})\,.
\end{aligned}$$

But $C_{N(y-\ell_2)} < C_N$, a contradiction to the minimality of C_N. If $\mathsf{v}_{\ell_1}(S)(\ell_1 - y) \ge \mathsf{v}_{\ell_2}(S)(y - \ell_2)$, then we can get a contradiction similarly.

Therefore $C_N \le 2$. Note that

$$\ell_2 = \min\{\ell_i \mid i \in [1, Nk]\} \le \frac{\sum_{i=1}^{Nk} \ell_i}{Nk} = \frac{\ell}{k} \le \max\{\ell_i \mid i \in [1, Nk]\} = \ell_1\,.$$

If $\ell_1 = \ell_2$, then $\ell/k = \ell_1$, a contradiction to $t_j < \ell/k < t_{j+1}$. Thus we get that $\ell_2 < \ell/k < \ell_1$. Since t_j is the maximal element of I that is smaller than ℓ/k and t_{j+1} is the minimal element of I that is larger than ℓ/k, we obtain that $\ell_2 \le t_j < t_{j+1} \le \ell_1$ whence $\{t_j, t_{j+1}\} \subset I_M \cap [\ell_2, \ell_1]$. Since $C_N \le 2$ and $\{\ell_i \mid i \in [1, Nk]\} \subset I \cap [\ell_2, \ell_1]$, it follows that

$$\{\ell_i \mid i \in [1, Nk]\} = I \cap [\ell_2, \ell_1] = \{t_j, t_{j+1}\}\,. \tag{3.1}$$

Then

$$\begin{aligned}
\rho(a^{Nk} b^{N\ell}) &= \frac{\max \mathsf{L}(a^{Nk} b^{N\ell})}{\min \mathsf{L}(a^{Nk} b^{N\ell})} \\
&= \frac{\sum_{i=1}^{Nk} \max \mathsf{L}(ab^{\ell_i})}{\sum_{i=1}^{Nk} \min \mathsf{L}(ab^{\ell_i})} \\
&= \frac{x_1 \max \mathsf{L}(ab^{t_j}) + x_2 \max \mathsf{L}(ab^{t_{j+1}})}{x_1 \min \mathsf{L}(ab^{t_j}) + x_2 \min \mathsf{L}(ab^{t_{j+1}})}\,,
\end{aligned}$$

where by (3.1)

$$x_1 = |\{i \in [1, Nk] \mid \ell_i = t_j\}| \quad \text{and} \quad x_2 = |\{i \in [1, Nk] \mid \ell_i = t_{j+1}\}| .$$

Comparing exponents of a and b we obtain the equations

$$x_1 t_j + x_2 t_{j+1} = N\ell \quad \text{and} \quad x_1 + x_2 = Nk$$

whence

$$(x_1, x_2) = \left(\frac{(kt_{j+1} - \ell)N}{t_{j+1} - t_j}, \frac{(\ell - kt_j)N}{t_{j+1} - t_j} \right) .$$

Plugging in this expression for (x_1, x_2) we obtain that

$$\begin{aligned} \rho(a^k b^\ell) = \rho(a^{Nk} A_0^{N\ell}) &= \frac{(kt_{j+1} - \ell) \max \mathsf{L}(ab^{t_j}) + (\ell - kt_j) \max \mathsf{L}(ab^{t_{j+1}})}{(kt_{j+1} - \ell) \min \mathsf{L}(ab^{t_j}) + (\ell - kt_j) \min \mathsf{L}(ab^{t_{j+1}})} \\ &= \frac{(t_{j+1} - x) \max \mathsf{L}(ab^{t_j}) + (x - t_j) \max \mathsf{L}(ab^{t_{j+1}})}{(t_{j+1} - x) \min \mathsf{L}(ab^{t_j}) + (x - t_j) \min \mathsf{L}(ab^{t_{j+1}})} . \end{aligned}$$

5. It follows by Lemma 3.1.2 that $\overline{R}(H) \subset \{q \in \mathbb{Q} \mid \inf \overline{R}(H) \le q \le \sup \overline{R}(H)\}$.

By 3., we know that $\rho(ab^{t_j}) \in \overline{R}(H)$ for each $j \in \mathbb{N}$ and $\rho(H) \in \overline{R}(H)$. If suffices to prove that $\{q \in \mathbb{Q} \mid \rho(ab^{t_{j+1}}) < q < \rho(ab^{t_j}))\} \subset \overline{R}(H)$.

By 4., we obtain that for each $j \in \mathbb{N}$,

$$\begin{aligned} &\{q \in \mathbb{Q} \mid \rho(ab^{t_{j+1}}) < q < \rho(ab^{t_j}))\} \\ =&\left\{ \frac{(t_{j+1} - x) \max \mathsf{L}(ab^{t_j}) + (x - t_j) \max \mathsf{L}(ab^{t_{j+1}})}{(t_{j+1} - x) \min \mathsf{L}(ab^{t_j}) + (x - t_j) \min \mathsf{L}(ab^{t_{j+1}})} \mid x \in \mathbb{Q} \text{ and } t_j < x < t_{j+1} \right\} \\ \subset&\{\rho(a^k b^\ell) \mid (k, \ell) \text{ is a nice pair with respcet to } (a, b) \text{ such that } t_j < \ell/k < t_{j+1}\} \\ \subset&\overline{R}(H) . \end{aligned}$$

□

Proof of Theorem 1.1. Suppose that H is a locally finitely generated monoid. For every $q \in \mathbb{Q}$ with $\sup \overline{R}(H) > q > \inf \overline{R}(H)$, there exist $a, b \in H \setminus H^\times$ such that $\lim_{n\to\infty} \rho(a^n) > q > \lim_{n\to\infty} \rho(b^n)$.

Let $S = [\![ab]\!]$. Then S_{red} is finitely generated. It follows by Proposition 3.4.5 that

$$q \in \left\{p \in \mathbb{Q} \mid \lim_{n\to\infty} \rho(a^n) > p > \lim_{n\to\infty} \rho(b^n)\right\} \subset \overline{R}(S) \subset \overline{R}(H) .$$

Therefore H is asymptotic fully elastic.

If $\min \overline{R}(H) = 1$, then it follows by Proposition 3.4.5 and Lemma 3.1.2 that

$$\overline{R}(H) = \{q \in \mathbb{Q} \mid 1 \le q \le \rho(H)\} \subset \{\rho(L) \mid L \in \mathcal{L}(H)\} .$$

Therefore H is fully elastic. □

Proof of Theorem 1.2. Without restriction we may suppose that H is reduced and we set $\mathcal{A}(H) = \{u_1, \ldots, u_m\}$. Proposition 3.4.5 and Lemma 3.1.2 imply that

$$\overline{R}(H) = \{q \in \mathbb{Q} \mid 1 \le q \le \rho(H)\} \subset \{\rho(L) \mid L \in \mathcal{L}(H)\}\,.$$

Thus it remains to prove that, for a given $\epsilon > 0$, the set

$$\{\rho(L) \mid L \in \mathcal{L}(H) \text{ and } 1 \le \rho(L) \le r - \epsilon\}$$

is finite. Assume to the contrary that this set is infinite. Then there is a sequence $(a_k)_{k=1}^{\infty}$ with elements from H such that $1 \le \rho(a_k) \le r - \epsilon$ and $(\rho(a_k))_{k=1}^{\infty}$ is strictly increasing or decreasing. For each k, we fixed a factorization $z_k = u_1^{t_k^{(1)}} \cdot \ldots \cdot u_m^{t_k^{(m)}}$ of a_k with $|z_k| = \min \mathsf{L}(a_k)$. Let $I \subset [1,m]$ be the set of all $i \in [1,m]$ such that $\{t_k^{(i)} \mid k \in \mathbb{N}\}$ is finite and let $M \in \mathbb{N}$ such that $t_k^{(i)} \le M$ for all $k \in \mathbb{N}$ and $i \in I$. For each $j \in [1,m] \setminus I$, let $\rho_j = \lim_{n \to \infty} \rho(u_j^n)$ and let $N \in \mathbb{N}$ such that $\rho_j = \rho(u_j^N)$ for all $j \in [1,m] \setminus I$. Then for all $k \ge N$, we have $\max \mathsf{L}(u_j^k) = k\rho_j$ for all $j \in [1,m] \setminus I$ by the minimality of $|z_k|$ and $\{t_k^{(j)}\}$ is infinity.

It follows that

$$\rho(a_k) \ge \frac{\sum_{j \in [1,m] \setminus I} t_k^{(j)} \rho_j}{M(m - |I|) + \sum_{j \in [1,m] \setminus I} t_k^{(j)}}$$

and hence

$$\lim_{k \to \infty} \rho(a_k) \ge \lim_{k \to \infty} \frac{\sum_{j \in [1,m] \setminus I} t_k^{(j)} \rho_j}{\sum_{j \in [1,m] \setminus I} t_k^{(j)}} \ge r\,,$$

a contradiction. $\square$

Proposition 3.5. *If $(H_i)_{i \in \mathbb{N}}$ is a family of* BF*-monoids with the same elasticity, then their coproduct $\coprod_{i=1}^{\infty} H_i$ is fully elastic.*

Proof. Let $H = \coprod_{i=1}^{\infty} H_i$ and $r = \rho(H_1)$. It is clear that $\rho(H) \ge r$. Let $a \in H$. Then there exist a finite subset $I \subset \mathbb{N}$ and $b_i \in H_i$ for each $i \in I$ such that $a = \prod_{i \in I} b_i$. Therefore

$$\mathsf{L}(a) = \sum_{i \in I} \mathsf{L}(b_i)$$

and hence

$$\rho(a) = \frac{\max \mathsf{L}(b_i)}{\min \mathsf{L}(b_i)} \le \max\{\rho(b_i) \mid i \in I\} \le r\,.$$

It follows that $\rho(H) = r$.

Let $q = \frac{m}{n} \in \mathbb{Q}$ with $1 < q < \rho(H)$, where $m, n \in \mathbb{N}$. Then for each $i \in \mathbb{N}$, there exists $a_i \in H_i$ such that $\rho(a_i) = \frac{m_i}{n_i} \ge q$, where $m_i, n_i \in \mathbb{N}$. There exist $d \in [0, m-n-1]$ and $J \subset \mathbb{N}$ with $|J| = m - n$ such that $m_j - n_j \equiv d \pmod{m-n}$. Therefore $m - n \mid \sum_{j \in J}(m_j - n_j)$. Since $\frac{\sum_{j \in J} m_j}{\sum_{j \in J} n_j} \ge \frac{m}{n}$, we have

$$x = \frac{n \sum_{j \in J} m_j - m \sum_{j \in J} n_j}{m - n} = \frac{n \sum_{j \in J}(m_j - n_j) - (m-n) \sum_{j \in J} n_j}{m - n} \in \mathbb{N}\,.$$

It follows that $q = \frac{x+\sum_{j\in J} m_j}{x+\sum_{j\in J} n_j}$

Let $J_1 \subset \mathbb{N}$ with $|J_1| = x$, $J_1 \cap J = \emptyset$ and let $c_i \in \mathcal{A}(H_i)$ for each $i \in J_1$. Therefore $\rho(\prod_{j\in J} b_j \prod_{i\in J_1} c_i) = \frac{x+\sum_{j\in J} m_j}{x+\sum_{j\in J} n_j} = q$ and hence H is fully elastic. $\square$

Example 3.6. Let H be a finitely generated monoid with $\min \overline{R}(H) = r > 1$ (Note that numerical monoids, distinct from $(\mathbb{N}_0, +)$, have this property and they are not fully elastic). Then the coproduct $H^* = \coprod_{i=1}^{\infty} H_i$, with $H_i = H$ for all $i \in \mathbb{N}$, is locally finitely generated. Since $\inf \overline{R}(H^*) = \inf\{\min \overline{R}(H_i) \mid i \in \mathbb{N}\}$, it follows that $\min \overline{R}(H^*) = r$. By Proposition 3.5, we infer that H^* is fully elastic.

4. Weakly Krull monoids

Let H be a monoid. We denote by $\mathfrak{X}(H)$ the set of minimal nonempty prime s-ideals of H, and by $\mathfrak{m} = H \setminus H^\times$ the maximal s-ideal. Let $\mathcal{I}_v^*(H)$ denote the monoid of v-invertible v-ideals of H (with v-multiplication). Then $\mathcal{F}_v(H)^\times = \mathsf{q}(\mathcal{I}_v^*(H))$ is the quotient group of fractional v-invertible v-ideals, and $\mathcal{C}_v(H) = \mathcal{F}_v(H)^\times / \{xH \mid x \in \mathsf{q}(H)\}$ is the v-class group of H (detailed presentations of ideal theory in commutative monoids can be found in [22, 17]).

We denote by

- $\widehat{H} = \{a \in \mathsf{q}(H) \mid$ there exists $c \in H$ such that $ca^n \in H$ for all $n \in \mathbb{N}\}$ the completely integral closure of H,
- $(H : \widehat{H}) = \{x \in \mathsf{q}(H) \mid x\widehat{H} \subset H\} \subset H$ the conductor of H, and
- $H' = \{a \in \mathsf{q}(H) \mid$ there exists $N \in \mathbb{N}$ such that $a^n \in H$ for all $n \ge N\}$ the seminormal closure of H.

We say H is completely integrally closed if $H = \widehat{H}$ and H is seminormal if $H' = H$.

To start with the local case, we recall that H is said to be

- *primary* if $\mathfrak{m} \ne \emptyset$ and for all $a, b \in \mathfrak{m}$ there is an $n \in \mathbb{N}$ such that $b^n \subset aH$.
- *strongly primary* if $\mathfrak{m} \ne \emptyset$ and for every $a \in \mathfrak{m}$ there is an $n \in \mathbb{N}$ such that $\mathfrak{m}^n \subset aH$. We denote by $\mathcal{M}(a)$ the smallest n having this property.
- *finitely primary* if there exist $s, \alpha \in \mathbb{N}$ such that H is a submonoid of a factorial monoid $F = F^\times \times [q_1, \ldots, q_s]$ with s pairwise non-associated primes $q_1, \ldots, q_s$ satisfying
$$H \setminus H^\times \subset q_1 \cdot \ldots \cdot q_s F \text{ and } (q_1 \cdot \ldots \cdot q_s)^\alpha F \subset H\,.$$
- a *discrete valuation monoid* if it is primary and contains a prime element (equivalently, $H_{\mathrm{red}} \cong (\mathbb{N}_0, +)$).

Every strongly primary monoid is a primary BF-monoid ([17, Section 2.7]). If H is finitely primary, then H is strongly primary, $F = \widehat{H}$, $|\mathfrak{X}(H)| = s$ is called the rank of H, and H is seminormal if and only if
$$H = H^\times \cup q_1 \cdot \ldots \cdot q_s F\,.$$

Theorem 4.1. *Let H be a strongly primary monoid that is not half-factorial.*

1. *H is asymptotic fully elastic. More precisely, $\overline{R}(H) = \{\rho(H)\}$.*
2. *$\rho(H)$ is the only limit point of the set $\{\rho(L) \mid L \in \mathcal{L}(H)\}$.*
3. *If H is a seminormal finitely primary monoid with rank ≥ 2, then $\{\rho(L) \mid L \in \mathcal{L}(H)\} = \{\frac{n}{2} \mid n \in \mathbb{N}_{\ge 2}\}$.*

Proof. We start with the following claim.

Claim A. Let $(a_k)_{k=1}^{\infty}$ be a sequence with $a_k \in H$ such that $\lim_{k\to\infty} \max \mathsf{L}(a_k) = \infty$. Then $\liminf_{k\to\infty} \rho(a_k) \ge \rho(H)$.

Proof of **Claim A.** Let $b \in H \setminus H^\times$. By definition, for each $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $b^{n_k} \mid a_k$ and $b^{n_k+1} \nmid a_k$. Suppose $a_k = b^{n_k} b_k$ for each $k \in \mathbb{N}$, where $b_k \in H$. Since H is strongly primary, we infer that $\max \mathsf{L}(b_k) < \mathcal{M}(a)$. It follows by $\lim_{k\to\infty} \max \mathsf{L}(a_k) = \infty$ that $\lim_{k\to\infty} n_k = \infty$. Therefore

$$\rho(a_k) \ge \frac{n_k \max \mathsf{L}(b) + \max \mathsf{L}(b_k)}{n_k \min \mathsf{L}(b) + \min \mathsf{L}(b_k)} \ge \frac{n_k \max \mathsf{L}(b) + \mathcal{M}(a)}{n_k \min \mathsf{L}(b) + \mathcal{M}(a)}$$

and hence $\liminf_{k\to\infty} \rho(a_k) \ge \rho(b)$. The assertion follows immediately. $\square$[End of **Claim A.**]

1. We only need to prove $\overline{R}(H) \subset \{\rho(H)\}$. Let $a \in H \setminus H^\times$. Then $\lim_{k\to\infty} \max \mathsf{L}(a^k) = \infty$. By **Claim A.**, $\overline{\rho}(a) = \lim_{k\to\infty} \rho(a^k) \ge \rho(H)$. It follows by $\sup \overline{R}(H) \le \rho(H)$ that $\overline{R}(H) = \{\rho(H)\}$.

2. Assume to the contrary that there is another limit point $r \in \mathbb{R}$ with $1 \le r < \rho(H)$. Then there exists a sequence $(a_k)_{k=1}^{\infty}$ such that $\lim_{k\to\infty} \rho(a_k) = r$ and $\lim_{k\to\infty} \max \mathsf{L}(a_k) = \infty$. It follows by **Claim A.** that $r = \lim_{k\to\infty} \rho(a_k) \ge \rho(H)$, a contradiction to $r < \rho(H)$.

3. Let H be a seminormal finitely primary monoid with rank $r \ge 2$. Then $H = H^\times \cup q_1 \cdot \ldots \cdot q_s \widehat{H}$, where $q_1, \ldots, q_s$ are s pairwise non-associated primes of $\widehat{H}$.

Let $a \in H$. If a is an atom or a unit, then $\rho(a) = 1 = \frac{2}{2}$. Otherwise $a = \epsilon q_1^{k_1} \cdot \ldots \cdot q_s^{k_s}$ with $\epsilon \in \widehat{H}^\times$ and $k_1, \ldots, k_s \in \mathbb{N}_{\ge 2}$. Since $q_1 q_2^{k_2-1} \cdot \ldots \cdot q_s^{k_s-1}$ and $\epsilon q_1^{k_1-1} q_2 \cdot \ldots \cdot q_s$ are atoms, it follows that $2 \in \mathsf{L}(a)$ and $\rho(a) = \frac{\max \mathsf{L}(a)}{2}$. Thus $\{\rho(L) \mid L \in \mathcal{L}(H)\} \subset \{\frac{n}{2} \mid n \in \mathbb{N}_{\ge 2}\}$.

Let $n \in \mathbb{N}_{\ge 2}$ and $a = (q_1 \cdot \ldots \cdot q_s)^n$. Then $\min \mathsf{L}(a) = 2$ and $\max \mathsf{L}(a) = n$ which imply that $\rho(a) = \frac{n}{2}$. Therefore $\{\rho(L) \mid L \in \mathcal{L}(H)\} = \{\frac{n}{2} \mid n \in \mathbb{N}_{\ge 2}\}$. $\square$

We say H is *weakly Krull* ([22, Corollary 22.5]) if $H = \bigcap_{\mathfrak{p} \in \mathfrak{X}(H)} H_{\mathfrak{p}}$ and $\{\mathfrak{p} \in \mathfrak{X}(H) \mid a \in \mathfrak{p}\}$ is finite for all $a \in H$ and H is *weakly factorial* if one of the following equivalent conditions is satisfied ([22, Exercise 22.5]):

- Every non-unit is a finite product of primary elements.
- H is a weakly Krull monoid with trivial t-class group.

Clearly, every localization $H_{\mathfrak p}$ of H at a minimal prime ideal $\mathfrak p \in \mathfrak X(H)$ is primary, and a weakly Krull monoid H is v-noetherian if and only if $H_{\mathfrak p}$ is v-noetherian for each $\mathfrak p \in \mathfrak X(H)$. Every v-noetherian primary monoid is strongly primary and v-local ([18, Lemma 3.1]), and every strongly primary monoid is a primary BF-monoid ([17, Section 2.7]). Therefore the coproduct of a family of strongly primary monoids is a BF-monoid, and every coproduct of a family of primary monoids is weakly factorial. A v-noetherian weakly Krull monoid H is weakly factorial if and only if $\mathcal C_v(H) = 0$ if and only if $H_{\rm red} \cong \mathcal I_v^*(H)$.

Let R be a domain. We say R is a Mori domain, if $R^\bullet$ is v-noetherian. R is weakly Krull (resp. weakly factorial) if and only if its multiplicative monoid $R^\bullet$ is weakly Krull (resp. weakly factorial). Weakly Krull domains were introduced by D. D. Anderson, D. F. Anderson, Mott, and Zafrullah ([2, 4]). We recall some most basic facts. The monoid $R^\bullet$ is primary if and only if R is one-dimensional and local. If R is a one-dimensional local Mori domain, then $R^\bullet$ is strongly primary([17, Proposition 2.10.7]). Furthermore, every one-dimensional semilocal Mori domain with nontrivial conductor is weakly factorial and the same holds true for generalized Cohen-Kaplansky domains. It can be seen from the definition that one-dimensional noetherian domains are v-noetherian weakly Krull domains.

We continue with T-block monoids which are weakly Krull monoids of a combinatorial flavor and are used to model general weakly Krull monoids. Let G be an additive abelian group, $G_0 \subset G$ a subset, T a reduced monoid and $\iota \colon T \to G$ a homomorphism. Let $\sigma \colon \mathcal F(G_0) \to G$ be the unique homomorphism satisfying $\sigma(g) = g$ for all $g \in G_0$. Then

$$B = \mathcal B(G_0, T, \iota) = \{ S\, t \in \mathcal F(G_0) \times T \mid \sigma(S) + \iota(t) = 0 \} \subset \mathcal F(G_0) \times T = F$$

the *T-block monoid over G_0 defined by ι*. For details about T-block monoids, see [19, Section 4].

Let D be another monoid. A homomorphism $\phi : H \to D$ is said to be

- divisor homomorphism if $\phi(u) \mid \phi(v)$ implies that $u \mid v$ for all $u, v \in H$.
- cofinal if for every $a \in D$, there exists $u \in H$ such that $a \mid \phi(u)$.

Suppose $\phi : H \to D$ is a divisor homomorphism and D is reduced. Then $\phi(H)$ is a saturated submonoid of D and $H_{\rm red}$ is isomorphic to $\phi(H)$(see [17, Proposition 2.4.2.4]). Therefore we can view $H_{\rm red}$ as a saturated submonoid of D.

The following is the main theorem of this section. Clearly, orders R in algebraic number fields satisfy all assumptions of Theorem 4.2. In particular, $\mathcal C_v(R)$ is finite, it coincides with the Picard group, and every class contains a regular prime ideal (whence $G_{\mathcal P} = \mathcal C_v(H)$ holds). Suppose R is a v-noetherian weakly Krull domain with nonzero conductor. Then all localizations $R_{\mathfrak p}$ are finitely primary and if R is not semilocal, then R has a regular element with is not a unit whence $\mathcal B(G_{\mathcal P}) \ne \{1\}$. For an extended list of examples, we refer to [19, Examples 5.7].

Theorem 4.2. *Let H be a v-noetherian weakly Krull monoid with the conductor $\emptyset \ne \mathfrak f = (H : \widehat H) \subsetneq H$ such that the localization $H_{\mathfrak p}$ is finitely primary for each minimal prime ideal $\mathfrak p \in \mathfrak X(H)$. We set $\mathcal P^* = \{\mathfrak p \in \mathfrak X(H) \mid \mathfrak p \supset \mathfrak f\}$ and suppose $\mathcal P = \mathfrak X(H) \setminus \mathcal P^* \ne \emptyset$. Let $G_{\mathcal P} \subset \mathcal C_v(H)$ denote the set of classes containing a minimal prime ideal from $\mathcal P$, and let $\pi : \mathfrak X(\widehat H) \to \mathfrak X(H)$ be the natural map defined by $\pi(\mathfrak P) = \mathfrak P \cap H$ for all $\mathfrak P \in \mathfrak X(\widehat H)$.*

1. *Suppose π is bijective. If $\widehat{H_{\mathfrak p}}^\times/H_{\mathfrak p}^\times$ is finite for each minimal prime ideal $\mathfrak p\in\mathfrak X(H)$ and $\mathcal B(G_{\mathcal P})\ne\{1\}$, then H is fully elastic.*

2. *Suppose π is not bijective. If H is seminormal and $G_{\mathcal P}=\mathcal C_v(H)$ is finite, then $\rho(H)=\infty$ and H is fully elastic.*

Proof. Let $\delta_H\colon H\to\mathcal I_v^*(H)$ be the homomorphism defined by $\delta_H(a)=aH$. Then δ_H is a cofinal divisor homomorphism with $\mathcal C(\delta_H)=\mathcal C_v(H)$ by [17, Proposition 2.4.5].

Since H is v-noetherian, we obtain $\mathcal P^*$ is finite and non-empty by [17, Theorem 2.2.5.1], say $\mathcal P^*=\{\mathfrak p_1,\ldots,\mathfrak p_n\}$ with $n\in\mathbb N$. By [17, Theorem 2.6.5.3] that $H_{\mathfrak p}$ ia a discrete valuation monoid for each $\mathfrak p\in\mathcal P$ and by [19, Proposition 5.3.4], there exists an isomorphism

$$\chi\colon\mathcal I_v^*(H)\to D=\coprod_{\mathfrak p\in\mathfrak X(H)}(H_{\mathfrak p})_{\rm red}=\mathcal F(\mathcal P)\times(H_{\mathfrak p_1})_{\rm red}\times\ldots\times(H_{\mathfrak p_n})_{\rm red}$$

where $\chi\,|\,\mathcal P=\mathrm{id}_{\mathcal P}$ and, for all $i\in[1,n]$, $D_i:=(H_{\mathfrak p_i})_{\rm red}$ is a reduced finitely primary monoid. Hence $\chi\circ\delta_H\colon H\to D$ is a cofinal divisor homomorphism with $\mathcal C(\chi\circ\delta_H)\cong\mathcal C_v(H)$ and we can view $H_{\rm red}$ as a saturated submomoid of D.

By [17, Proposition 3.4.8], it is sufficient to prove the assertions for the associated T-block monoid

$$B=\mathcal B(G_{\mathcal P},T,\iota)\subset F=\mathcal F(G_{\mathcal P})\times T\,,$$

where $T=D_1\times\ldots\times D_n$ and $\iota\colon T\to G$ is defined by $\iota(t)=[t]\in\mathcal C_v(H)$ for all $t\in T$.

1. Since π is bijective, it follows by [19, Lemma 5.1.3] that $H_{\mathfrak p}$ has rank 1 for each $\mathfrak p\in\mathfrak X(H)$. Since $\widehat{H_{\mathfrak p_i}}^\times/H_{\mathfrak p_i}^\times$ is finite for each $i\in[1,n]$, we have D_i is finitely generated for each $i\in[1,n]$. It follows that F is locally finitely generated. Therefore B is locally finitely generated.

Since $\mathcal B(G_{\mathcal P})\ne\{1\}$ implies that there exists a half-factorial subset $G_1\subset G_P$ such that $\mathcal B(G_1)\ne\{1\}$ by [20, Lemma 5.4], it follows that $\min\overline R(\mathcal B(G_{\mathcal P}))=1$. Hence $\min\overline R(B)=1$ and B is fully elastic by Theorem 1.

2. Since π is not bijective, it follows by [19, Lemma 5.1.3] that there exists $i\in[1,n]$ such that the rank s_i of $H_{\mathfrak p_i}$ is larger than 1, say i=1. Then D_1 is a seminormal reduced finitely primary monoid of rank $s\ge 2$. Suppose $\widehat{D_1}=\widehat{D_1}^\times[p_1,\ldots,p_s]$, where $p_1,\ldots,p_s$ are s pairwise non-associated prime elements. Then $D_1=p_1\cdot\ldots\cdot p_s\widehat D\cup\{1\}$. Let $a=\epsilon p_1^{\alpha_1}\cdot\ldots\cdot p_s^{\alpha_s}\in D_1$ with $\epsilon\in\widehat{D_1}^\times$ and $a\in\mathcal A(B)$ such that $|a|=\alpha_1+\ldots+\alpha_s$ is minimal and let $n=|G_{\mathcal P}|$.

If there exists $\delta\in\widehat{D_1}^\times$ such that $b=\delta p_1\cdot\ldots\cdot p_s\in B$, then $b\in\mathcal A(B)$. We infer that $c_k=\delta p_1p_2^{kn+1}\cdot\ldots\cdot p_s^{kn+1}$ and $d_k=\delta^{kn+1}p_1^{kn+1}p_2\cdot\ldots\cdot p_s$ are atoms of B for all $k\in\mathbb N$. Since $\max\mathsf L(c_kd_k)=kn+2$, we obtain that $\rho(c_kd_k)=\frac{kn+2}{2}$ for all $k\in\mathbb N$ and $\max\mathsf L(c_kd_k)-\min\mathsf L(c_kd_k)=kn$.

Otherwise let $\iota(p_1\cdot\ldots\cdot p_s)=g\ne 0$. Then $p_1\cdot\ldots\cdot p_s(-g)$ is an atom of B. We infer that $c_k=p_1p_2^{k\alpha_2n+1}\cdot\ldots\cdot p_s^{k\alpha_sn+1}(-g)$ and $d_k=p_1^{k\alpha_1n+1}p_2\cdot\ldots\cdot p_s(-g)$ are atoms of B for all $k\in\mathbb N$. Since $\max\mathsf L(c_kd_k)=kn+2$, we obtain that $\rho(c_kd_k)=\frac{kn+2}{2}$ for all $k\in\mathbb N$ and $\max\mathsf L(c_kd_k)-\min\mathsf L(c_kd_k)=kn$.

Thus in both cases there is a sequence $(a_k)_{k=1}^\infty$ with term $a_k\in H$ such that $\rho(a_k)=(kn+2)/2$ and $\max\mathsf L(a_k)-\min\mathsf L(a_k)=kn$. Therefore $\rho(H)=\infty$.

Since the sequence $0_{G_{\mathcal{P}}} \in \mathcal{F}(G_{\mathcal{P}})$ is a prime element of B, it follows by [9, Lemma 2.11] that

$$\begin{aligned} &\bigcup_{k=1}^{\infty} \{a/b \in \mathbb{Q} \mid 1 \le a/b \le \rho(a_k), a-b \mid \max \mathsf{L}(a_k) - \min \mathsf{L}(a_k)\} \\ =&\bigcup_{k=1}^{\infty} \{a/b \in \mathbb{Q} \mid 1 \le a/b \le \frac{kn+2}{2}, a-b \,|\, kn\} \\ \subset&\{\rho(L) \mid L \in \mathcal{L}(H)\}\,. \end{aligned}$$

Let $q = a/b \in \mathbb{Q}$ with $q \ge 1$, where $a, b \in \mathbb{N}$ with $\gcd(a,b) = 1$. Choose $k_0 = 2a(a-b)$. Then $a/b \le k_0/2 \le (k_0 n + 2)/2$ and $a - b \,|\, k_0 n$. It follows that

$$q \in \{a/b \in \mathbb{Q} \mid 1 \le a/b \le \frac{k_0 n + 2}{2}, a-b \,|\, k_0 n\} \subset \{\rho(L) \mid L \in \mathcal{L}(H)\}\,.$$

Therefore H is fully elastic. $\square$

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